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On Contraction of Time-varying Port-Hamiltonian Systems

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Abstract

In this paper we identify classes of port-Hamiltonian systems which are contracting. Our motivation in this study is two-fold, on one hand, it is well-known that many physical systems are described by port-Hamiltonian models. On the other hand, contraction is a fundamental property that has been efficiently exploited for the design of observers, as well as tracking, adaptive and multi-agent controllers for nonlinear systems. The conditions for contraction are given in terms of feasibility of linear matrix inequalities, hence their verification is computationally efficient.

Keywords: Nonlinear Systems, Contraction Theory, port-Hamiltonian Systems

1. Introduction

In many practical applications, one is interested not in the behaviour of trajectories with respect to attractors, but instead in the stability of trajectories with respect to one another. A key property to study this relative stability of trajectories is *contraction*, that is, if all solutions of the system tend to each other at infinity. The concept was, apparently, first introduced by Demidovich in his 1961 paper [7]—see also [20] for an early reference and [10] for a historical appraisal of the seminal work of Demidovich. In view of its wide applicability in various modern control tasks, the interest in contraction analysis revived several decades after these publications. It was popularized in the control community by Lohmiller and Slotine [9] who, independently, obtained and extended the result of Demidovich. A Lyapunov approach unifying both state-space and input-to-output approaches to studying stability of solutions with respect to each other was developed by Angeli [4]. This approach is compatible with the input-to-state stability framework [14]. As it was pointed out in these papers, observer design and (controlled) synchronization problems are examples of possible applications of such stability properties, hence, it has attracted the attention of researchers during recent years—see, *e.g.*, [2, 3, 6, 8, 9, 14, 17, 18] for a series of applications and a review of the literature.

A key observation made in [9] is that the difficult question of defining a suitable error between trajectories, can be resolved invoking the integral of an infinitesimal measure of contraction. Replacing, in this way, the global concept of a distance between trajectories by a local one. Proofs of the relation between contraction of nonlinear, time-varying (NLTV) systems and the stability of its variational model are varied in style and its hard to put them in a unified context—see [8] for an attempt in this direction. To make our paper self contained we first prove that an NLTV system is exponentially contracting *if and only if* its variational model is uniformly globally exponentially stable (UGES)—a result that, although relatively easy to prove, we have not been able to find in the literature. This equivalence is particularly important, because it allows us to reduce the problem of analysis of contractivity to the classical problem of UGES of a linear time-varying (LTV) system, for which a wide range of Lyapunov-based tools are available in the literature—avoiding the need to checking existence of solutions of partial differential inequalities [9].

We are interested in the paper in the analysis of contraction of NLTV *port-Hamiltonian* (pH) systems. As shown in [15, 16], pH models describe the behavior of many physical processes and have

been extensively used in practical applications. In particular, it has been shown in [18] that contraction is a key property for the application of energy-shaping methods in *tracking* tasks—a problem that has remained open for many years. Invoking the aforementioned equivalence, we study contraction of pH systems applying standard results of stability of systems with local quadratic constraints, see [5] for a recent, tutorial reference. All the results are given in terms of feasibility of *linear matrix inequalities* (LMIs), hence they are computationally efficient.

Our main contributions may be summarized as follows.

- (i) We prove that a reasonable contraction conjecture is unfortunately wrong.
- (ii) For the case when the interconnection and damping matrices depend only on time, we give time-dependent LMI tests for contraction.
- (iii) The time dependence is removed from the LMI tests if these matrices leave in a polytope with constant matrix vertices. We give a second time-independent test, without the assumption of the matrices being in a polytope, but this time in terms of feasibility of a bi-linear matrix inequality.
- (iv) With some additional boundedness conditions on the Hamiltonian and the interconnection and damping matrices, the latter result using the polytope vertices is extended to the case when these matrices depend on the state and time.
- (v) It is shown that the recent characterization of contractive pH systems with constant interconnection and damping matrices reported in [18], which is given in term of absence of purely imaginary eigenvalues of a matrix, admits an equivalent LMI characterization, which turns out to be more-conservative than the LMI test given in the paper.

The remainder of the paper is organized as follows. In Section 2 we present the problem formulation and give a result on contraction. Section 3 is devoted to prove that a reasonable conjecture on contraction is, unfortunately, wrong. Section 4 considers the case when the interconnection and damping matrices depend only on time, while in Section 5 they depend on time and the state. The paper is wrapped-up with concluding remarks in Section 6.

Notation. We use $\mathbb{R}_+ := [0, \infty)$. For $x \in \mathbb{R}^n$, we denote the Euclidean norm $|x|^2 := x^\top x$, while $\|\cdot\|$ is a matrix norm. All mappings are assumed smooth and all dynamical systems are assumed to be forward complete. Given a mapping $H : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ we define its transposed gradient via the differential operator $\nabla_x H(x, t) := \left(\frac{\partial H}{\partial x}(x, t)\right)^\top$ and its Hessian as $\nabla_x^2 H(x, t) := \frac{\partial^2 H}{\partial x^2}(x, t)$. For a mapping $f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ we denote its Jacobian by $\nabla_x f(x, t) := \frac{\partial f}{\partial x}(x, t)$.

2. Contraction of NLTV Systems and Problem Formulation

In this section we present the contraction problem that we address in the paper and prove that it is equivalent to UGES of a variational model. To simplify the presentation, throughout the paper, we assume that all the properties hold globally, the extension to a local setting being straightforward.

2.1. Problem formulation

In this paper we consider NLTV port-Hamiltonian (pH) systems of the form [15]

$$\dot{x} = [\mathcal{J}(x, t) - \mathcal{R}(x, t)]\nabla_x H(x, t) \quad (1)$$

with $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ the interconnection matrix, $\mathcal{R} : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ the damping matrix verifying

$$\begin{aligned} \mathcal{J}(x, t) &= -\mathcal{J}^\top(x, t) \\ \mathcal{R}(x, t) &\geq 0, \end{aligned}$$

respectively, and $H : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ the Hamiltonian function.

To simplify the notation in the sequel we define the matrix $F : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ as $F(x, t) := \mathcal{J}(x, t) - \mathcal{R}(x, t)$ and the system (1) takes the form

$$\dot{x} = F(x, t)\nabla_x H(x, t) \quad (2)$$

where

$$F(x, t) + F^\top(x, t) \leq 0, \quad \forall x, t. \quad (3)$$

Notice, that (3) implies that all eigenvalues of matrices $F(x, t)$ have non positive real parts for all x, t . In the sequel we will require only that $F(x, t)$ is Hurwitz at least at one point (x, t) .

We are interested in investigating when the pH system (2) is *contracting* in the following sense.

Definition 1. Let $f: \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ and consider the system

$$\dot{x} = f(x, t). \quad (4)$$

The system (4) is called *contracting*, if there exists positive numbers ρ and m such that for all solutions $x_a(\cdot), x_b(\cdot)$ of (4) we have

$$|x_a(t) - x_b(t)| \leq me^{-\rho t}|x_a(0) - x_b(0)|, \quad \forall t \geq 0. \quad (5)$$

□□□

The key property is, of course, that all solutions of a contracting system tend to each other exponentially at infinity.

2.2. Motivation

Notice that we consider the case of *time-varying* Hamiltonian and damping and interconnection matrices. Our motivation to consider this case is threefold.

- (i) The analysis of the tracking controller for pH systems reported in [18], see also the recent work [12]. The current techniques of control of pH systems are, mainly, restricted to regulation tasks, and it is widely recognized that extending these results to handle tracking objectives is one of the biggest challenges in the field.
- (ii) The study of switched systems where the action of the switch, which may be the control signal $u(x, t)$, appears in the interconnection matrix as $\mathcal{J}(u(x, t))$ —a prototypical case of this class of systems is power converters, which are used in the seminal paper [8] as an illustration of contraction.
- (iii) The problem of stabilization of non-holonomic systems that, as is well-known, is achieved only with switching or time-varying controllers, as well as the design of controllers for orbital stabilization, that also leads to time-varying pH systems.

2.3. An equivalent characterization of contraction of NLTV systems

As indicated in the introduction many different ways to study contraction of a NLTV system have been reported in the literature. To make the paper self-contained we give the following characterization.

Definition 2. Let G be a set of time-varying matrices $g: [0, \infty) \rightarrow \mathbb{R}^{n \times n}$. The associated set of systems

$$\dot{\xi} = g(t)\xi, \quad (6)$$

is called uniformly globally exponentially stable (UGES), if there exist positive numbers C, β such that for every matrix $g \in G$ and every solution $\xi(\cdot)$ of system (6) the following inequality holds:

$$|\xi(t)| \leq Ce^{-\beta t}|\xi(0)|, \quad \forall t \geq 0.$$

Proposition 1. Assume $x(\cdot)$ is a solution of system (4). Consider a set of systems parameterised by solutions $x(\cdot)$ of system (4)

$$\dot{\xi} = \nabla_x f(x(t), t)\xi. \quad (7)$$

The system (4) is contracting if and only if (7) is UGES.

Proof. First, consider the proof of the “only if” statement. That is, if the system is contracting, then it should be UGES. To prove it we show that if the system is not UGES, then it should not be contracting. Negating the Definition 2 we get that the system (4) is not contracting if for all positive numbers C , β there exist solutions $x_a(\cdot)$, $x_b(\cdot)$ of system (4) and a number $T > 0$ such that

$$|x_a(T) - x_b(T)| > Ce^{-\beta T}|x_a(0) - x_b(0)|. \quad (8)$$

We need to show that this statement holds. Towards this end, we negate the Definition of UGES 1, yielding that, for every positive constants C_1 , β , there exists a solution $x(\cdot)$ of system (4) and a number $T > 0$ such that, for some $\xi(0)$,

$$|\xi(T)| > C_1e^{-\beta T}|\xi(0)|. \quad (9)$$

Now fix arbitrary positive numbers C and β . Find a solution $x(\cdot)$ and a number $T > 0$ corresponding to $C_1 = 2C$ and β . Denote by $\psi(t, x_0)$ the value $x(t)$ of the solution $x(\cdot)$ of system (4) with initial conditions $\psi(0, x_0) = x_0$. For every positive number δ denote

$$\xi_\delta(t) = \frac{\psi(t, x(0) + \delta\xi(0)) - \psi(t, x(0))}{\delta}.$$

Invoking well-known theorems of continuous dependence on the initial conditions of the solutions, we have that $\xi_\delta(t)$ tends to a solution of system (7) as $\delta \rightarrow 0$ uniformly on $t \in [0, T]$. Denote the limit by $\xi(t)$. Then $\xi(\cdot)$ satisfies inequality (9). In particular, for sufficiently small positive numbers δ we have

$$|\xi_\delta(T)| > \frac{2}{3}C_1e^{-\beta T}|\xi(0)| > Ce^{-\beta T}|\xi(0)|.$$

Now set

$$x_a(t) = \psi(t, x(0) + \delta\xi(0)), \quad x_b(t) = \psi(t, x(0)).$$

Then,

$$\begin{aligned} |x_a(0) - x_b(0)| &= \delta|\xi_\delta(0)|, \\ |x_a(T) - x_b(T)| &> \delta\frac{2}{3}C_1e^{-\beta T}|\xi(0)| > Ce^{-\beta T}|x_a(0) - x_b(0)|, \end{aligned}$$

which coincides with (8). Thus, according to the Definition 2, the system (4) is not contracting. This completes the “only if” part of the proposition.

In the proof of the “if” part Assume system (7) is UGES. Then, there exists positive constants C and β such that for all solutions $\xi(\cdot)$ of system (7)

$$|\xi(t)| \leq Ce^{-\beta t}|\xi(0)|, \quad \forall t \geq 0.$$

Invoking the general theorem about differentiability of solutions with respect to initial data given in Appendix A, we have that, for every $j = 1, \dots, n$, the partial derivatives $\eta_j(t) = \frac{\partial \psi}{\partial x_j}(t, x(0))$ are a solution of the system (7) with initial conditions $\eta_j(0) = e_j$, where e_j is the j th unit vector.

Due to the UGES assumption, all functions $\eta(\cdot)$ are uniformly exponentially stable, and therefore there exists positive constant C_1 and β such that, for all x_0 ,

$$\left\| \frac{\partial \psi}{\partial x}(t, x_0) \right\| \leq C_1e^{-\beta t}, \quad \forall t \geq 0.$$

Fix arbitrary vectors $x_a, x_b \in \mathbb{R}^n$. Then

$$\psi(t, x_a) - \psi(t, x_b) = \left[\int_0^1 \frac{\partial \psi}{\partial x}(t, sx_a + (1-s)x_b) ds \right] (x_a - x_b).$$

Hence,

$$|\psi(t, x_a) - \psi(t, x_b)| \leq C_1 e^{-\beta t} |x_a - x_b|,$$

and the system (4) is contracting, completing the proof. $\square\square\square$

Remark 1. In [9] it is stated that system (4) is contracting if and only if there exists a matrix $W : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ uniformly positive definite with respect to the first argument, which satisfies along all solutions $x(\cdot)$ of system (4) the initial conditions $W(x, 0) = cI$ and the following matrix differential equation:

$$\dot{W}(x(t), t) + W(x(t), t) \nabla_x f(x(t), t) + (\nabla_x f(x(t), t))^\top W(x(t), t) = -\beta W(x(t), t). \quad (10)$$

Actually

$$\dot{W} = \frac{\partial W}{\partial t} + \sum_{j=1}^n \frac{\partial W}{\partial x_j} f_j(x, t).$$

Therefore, (10) is a partial differential equation, whose existence of solutions on the half space $\mathbb{R}^n \times \mathbb{R}_+$ remains to be proven.

3. Disproving a Conjecture on Contraction of NLTV pH Systems

Given the particular structure of pH systems it seems reasonable to conjecture that if the Hamiltonian function is strongly convex and the matrix $F(x, t)$ is “asymptotically stable” for all frozen values of x and t , then the system will be contracting.¹ More precisely, we conjecture that the conditions

$$(i) \quad \Re\{\lambda_i(F(x, t))\} \leq \alpha < 0, \quad \forall x, t, \quad (11)$$

$$(ii) \quad 0 < c_1 I \leq \nabla_x^2 H(x, t) \leq c_2 I, \quad \forall x, t, \quad (12)$$

are sufficient to ensure that the pH system (2) is contracting. Unfortunately, the proposition below proves that the conjecture is wrong. Actually, it is shown that the two conditions above do not even guarantee boundedness of trajectories.

Proposition 2. There exist functions $F(x, t)$, $H(x, t)$ satisfying (11) and (12) such that the pH system (2) admits the zero trajectory and an unbounded trajectory, which allows us to conclude that the system is not contracting.

Proof. Set

$$F(x, t) = \begin{bmatrix} -1 & 2\sqrt{a(t)} \\ 0 & -a(t) \end{bmatrix},$$

$$H(x, t) = \frac{1}{2} x^\top Q(t) x, \quad Q(t) := \begin{bmatrix} 2\sqrt{a(t)} & 1 \\ 1 & \frac{1}{\sqrt{a(t)}} \end{bmatrix},$$

¹This conjecture was communicated to us by Prof. Yazdanpanah.

where $a(t) \geq \gamma > 0$ will be chosen later. It will be T -periodic and piecewise constant. It is easy to check that conditions (11) and (12) are satisfied.

System (2) is LTV of the form

$$\dot{x} = A(t)x, \quad (13)$$

where the matrix $A(t) := F(t)Q(t)$ has the following form

$$A(t) = \begin{bmatrix} 0 & 1 \\ -a(t) & -\sqrt{a(t)} \end{bmatrix}.$$

The function $x(t) \equiv 0$ is, obviously, a solution of the system (13). Hence, to disprove contraction it is sufficient to show that there exists a solution which does not tend to zero at infinity.

Assume $a(t) \equiv a_1$ on an interval $[0, t_1]$. We choose positive numbers a_1 and t_1 later. The eigenvalues of the matrix $A(t)$ are $\sqrt{a_1} \frac{-1 \pm i\sqrt{3}}{2}$. Consider a solution $x(\cdot)$ starting at a point

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is straightforward to compute

$$x(t) = e^{-\frac{\sqrt{a_1}}{2}t} \begin{bmatrix} \frac{2}{\sqrt{3a_1}} \sin(\frac{\sqrt{3a_1}}{2}t) \\ \cos(\frac{\sqrt{3a_1}}{2}t) - \frac{1}{\sqrt{3}} \sin(\frac{\sqrt{3a_1}}{2}t) \end{bmatrix}.$$

Denote by t_1 the first root of the second component of $x(t)$, that is,

$$t_1 = \frac{2\pi}{3\sqrt{3a_1}}.$$

Then

$$x(t_1) = e^{-\frac{\pi}{3\sqrt{3}}} \begin{bmatrix} \frac{1}{\sqrt{a_1}} \\ 0 \end{bmatrix}.$$

Set $a(t) \equiv a_2$ on the interval $[t_1, t_1 + t_2]$, where positive numbers t_2 , a_2 will be chosen later. It is easy to compute

$$x(t + t_1) = e^{-\frac{\pi}{3\sqrt{3}}} \frac{1}{\sqrt{a_1}} e^{-\frac{\sqrt{a_2}}{2}t} \begin{bmatrix} \cos(\frac{\sqrt{3a_2}}{2}t) + \frac{1}{\sqrt{3}} \sin(\frac{\sqrt{3a_2}}{2}t) \\ -\frac{\sqrt{3a_2}}{2} \sin(\frac{\sqrt{3a_2}}{2}t) - \frac{\sqrt{a_2}}{2\sqrt{3}} \sin(\frac{\sqrt{3a_2}}{2}t) \end{bmatrix}.$$

Denote by t_2 the first root of the first component of $x(t + t_1)$. Then,

$$t_2 = \frac{4\pi}{3\sqrt{3a_2}},$$

and

$$x(t_2 + t_1) = -e^{-\frac{\pi}{\sqrt{3}}} \frac{\sqrt{a_2}}{\sqrt{a_1}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Set $T = t_1 + t_2$. Define function $a(t)$ as a T -periodic function. If

$$e^{-\frac{\pi}{\sqrt{3}}} \frac{\sqrt{a_2}}{\sqrt{a_1}} > 1,$$

then the solution $x(t)$ tends to infinity exponentially, and system (2) is not contracting. $\square\square\square$

Remark 2. This two-dimensional example shows that the contraction property may depend on the range of change (in time) of the matrix $F(x, t)$. To ensure the contraction property we need to impose certain additional constraints (for example, uniform bounds) on $F(x, t)$. Notice that if system (2), (3) is time-invariant, then $H(x)$ qualifies as a Lyapunov function. If the function $H(x)$ is strongly convex, which is guaranteed by (12), then every solution tends to a set of stationary points. Hence, the contraction property is reduced to GES—a reduction that is lost if the system is time-varying.

4. Contraction when Matrix F Depends on Time Only

In this section we assume the matrix F is time-varying, but *does not depend* on x . First, we prove contraction if a time-dependent LMI condition holds, uniformly in t . Second, assuming that the matrix $F(t)$ lives inside a polytope with constant vertices, we give an LMI condition that depends only on the vertex matrices. Finally, another time-independent test, but this time with a bilinear matrix inequality, is given. As a corollary of our first result, we give an LMI condition for pH systems with constant F that contains, as a particular case, the condition given in Theorem 4 of [18] that, to the best of our knowledge, is the only result on contraction of pH systems reported in the literature. Consequently, our result extends the realm of application of the controller proposed in [18].

To simplify the notation we introduce now four positive constants that will be used in the sequel

$$\alpha := \frac{c_2 - c_1}{c_1 + c_2}, \quad \gamma := \frac{1}{2}(c_1 + c_2), \quad \eta := 1 - \frac{c_1}{c_2}, \quad \mu := \frac{1}{2}(c_2 - c_1). \quad (14)$$

4.1. A time-dependent LMI condition

Proposition 3. Consider the system (2), with $F = F(t)$, that is, a function of time only. Assume (12) hold and the matrix $F(t)$ is Hurwitz for at least one t . If there exists $P = P^\top$ and a positive number ϵ such that for all $t \geq 0$ the following inequality holds

$$\begin{bmatrix} PF(t) + F^\top(t)P + \alpha^2 I + \epsilon I & PF(t) \\ F^\top(t)P & -I \end{bmatrix} \leq 0, \quad (15)$$

where α is defined in (14).

Then, the pH system (2) is contracting.

Proof. According to Proposition 1 it is sufficient to show that the system

$$\dot{\xi}(t) = F(t)\nabla_x^2 H(x(t), t)\xi(t) \quad (16)$$

is UGES. Fix a solution $x(\cdot)$ of system (2). Denote

$$\begin{aligned} G(t) &:= \nabla_x^2 H(x(t), t) - \gamma I \\ \zeta(t) &= G(t)\xi(t). \end{aligned}$$

Then, the system (16) can be rewritten in the form

$$\dot{\xi}(t) = \gamma F(t)\xi(t) + F(t)\zeta(t). \quad (17)$$

Consider the (time-dependent) LMI

$$\begin{bmatrix} QF(t)\gamma + F^\top(t)Q\gamma + \mu^2 I + \delta I & QF(t) \\ F^\top(t)Q & -I \end{bmatrix} \leq 0 \quad (18)$$

that we assume holds, uniformly in time, for some $Q = Q^\top$ and $\delta > 0$. The matrix Q is positive definite since $F(t)$ is Hurwitz for at least one value of t , and the (1, 1) block of (15) is negative definite. Inequality (18) implies

$$2a^\top Q(\gamma F(t)a + F(t)b) + \mu^2 |a|^2 - |b|^2 \leq -\delta |a|^2, \quad \forall a, b \in R^n.$$

Consider the standard Lyapunov function candidate $V(\xi) = \xi^\top Q \xi$ for the system (17). Using the inequality above, some simple calculations show that its time derivative, along trajectories of the system (17), verifies

$$\dot{V} + \mu^2 |\xi(t)|^2 - |\zeta(t)|^2 \leq -\delta |\xi(t)|^2. \quad (19)$$

Now, from (12) we have that

$$-\mu I \leq G(t) \leq \mu I,$$

which implies the inequality

$$\mu^2 |\xi(t)|^2 - |\zeta(t)|^2 \geq 0. \quad (20)$$

Using (20) in (19) we have that $\dot{V}(\xi(t)) \leq -\delta |\xi(t)|^2$. Hence, if (18) holds, the system (17) is exponentially stable. Since the values of Q and δ do not depend on the solution $x(\cdot)$, the exponential stability is uniform with respect to $x(\cdot)$.

To complete the proof we will now show that the LMI (18) is equivalent to (15). Towards this end, set $Q = P\gamma$. Then, inequality (18) takes the form

$$\begin{bmatrix} PF(t)\gamma^2 + F^\top(t)P\gamma^2 + \mu^2 I + \delta I & PF(t)\gamma \\ F^\top(t)P\gamma & -I \end{bmatrix} \leq 0. \quad (21)$$

Multiplying (21) from the left and from the right by $\text{diag}\{\frac{1}{\gamma}I, I\}$ we get (15) with

$$\epsilon := \frac{\delta}{\gamma^2}.$$

□□□

Remark 3. It is interesting to note that the condition (15) of Proposition 3 is equivalent to the existence of a matrix $P = P^\top$ such that for all $t \geq 0$

$$\|PF(t) + I\|^2 \leq 1 - \epsilon - \alpha^2.$$

Indeed, inequality (15) is equivalent to

$$PF(t) + F^\top(t)P + (\alpha^2 + \epsilon)I + PF(t)F^\top(t)P \leq 0.,$$

which in turn is equivalent to

$$(PF(t) + I)(PF(t) + I)^\top \leq (1 - \epsilon - \alpha^2)I.$$

4.2. Comparison with Theorem 4 of [18]

In Theorem 4 of [18] it is shown that the system (2) with *constant* F is contracting if (11) and (12) hold, and for some $\epsilon > 0$, the matrix

$$\begin{bmatrix} F & \eta FF^\top \\ -(\eta + \epsilon)I & -F^\top \end{bmatrix}$$

has no purely imaginary eigenvalues, where η is defined in (14).

It is possible to show that the aforementioned eigenvalue condition is equivalent to existence of $P = P^\top$ such that the LMI

$$\begin{bmatrix} PF + F^\top P + \eta^2 I & PF \\ F^\top P & -I \end{bmatrix} < 0,$$

holds. Comparing this LMI with (15) and noting that $\eta > \alpha$, we conclude that the condition given in Proposition 3 is *strictly weaker*. Notice also that we have replaced the assumption (12) by the strictly weaker requirement that $F(t)$ is Hurwitz for at least one t .

4.3. A constant LMI condition for systems with $F(t)$ inside a convex polytope

Now assume the matrix $F(t)$ belongs to a convex polytope with constant vertices F_1, \dots, F_m for all t . That is, we assume that there exists functions $c_i(t) \geq 0$, $i = 1, \dots, m$, such that

$$F(t) = \sum_{i=1}^m c_i(t) F_i, \quad \forall t$$

$$\sum_{i=1}^m c_i(t) = 1. \quad (22)$$

Proposition 4. Consider the system (2), with $F = F(t)$. Assume (12) and (22) hold and the matrix $F(t)$ is Hurwitz for at least one t . If there exists a constant matrix $P = P^\top$ such that

$$\begin{bmatrix} PF_j + F_j^\top P + \alpha^2 I & PF_j \\ F_j^\top P & -I \end{bmatrix} < 0, \quad \forall j = 1, \dots, m, \quad (23)$$

then the system (2) is contracting.

Proof. Since the left hand side of (15) is linear with respect to the matrix F , the inequalities (23) imply (15). The proof is completed invoking Proposition 3. $\square\square\square$

Remark 4. The LMIs (23) are very easy to verify and only require the knowledge of the vertices of the polytope and the bounds on the Hessian of the Hamiltonian.

4.4. A bilinear matrix inequality condition

Proposition 3 provides a condition for existence of Lyapunov function for system (17) in the form $\xi^\top Q \xi$ with a constant matrix Q . The next statement guarantees the existence of a quadratic Lyapunov function for system (17) with a time-varying matrix Q .

Proposition 5. Consider the system (2), with $F = F(t)$. Assume that (12) holds and matrix $F(t)$ is Hurwitz for at least one t . Assume there exist a positive number ϵ , $n \times n$ constant matrices $R, S, \Gamma = \Gamma^\top, P = P^\top$, with S Hurwitz, and for all $t \geq 0$ the following matrix inequalities hold:

$$\begin{pmatrix} F(t)F^\top(t) & -F(t) \\ -F^\top(t) & (\alpha^2 + \epsilon)I \end{pmatrix} \leq \begin{pmatrix} RR^\top & -S \\ -S^\top & \Gamma \end{pmatrix}, \quad (24)$$

and

$$\begin{pmatrix} PS + S^\top P + \Gamma & RS \\ S^\top R & -I \end{pmatrix} \leq 0. \quad (25)$$

Then, system (2) is contracting.

Proof. The matrix P is positive definite since S is Hurwitz, $\Gamma > 0$, and $PS + S^\top P + \Gamma < 0$. Using the notation introduced in the proof of Proposition 3, we have that the inequality (20) may be written as

$$\alpha^2 \gamma^2 |\xi|^2 - |\zeta|^2 \geq 0, \quad (26)$$

where γ is defined in (14).

Consider the differential Riccati equation

$$\frac{d}{dt} Q(t) = -\gamma [Q(t)F(t) + F^\top(t)Q(t) + (\alpha^2 + \epsilon)I + Q(t)F(t)F^\top(t)Q(t)] \quad (27)$$

with initial conditions $Q(0) = P + I$. Invoking the lemma given in Appendix A, we have that

$$Q(t) \geq P > 0, \quad \forall t \geq 0.$$

Define the Lyapunov function candidate $U(\xi, t) := \xi^\top Q(t)\xi$, whose derivative along the solutions of (17) yields

$$\begin{aligned}
\dot{U} &= 2\xi^\top(t)Q(t)[\gamma F(t)\xi(t) + F(t)\zeta(t)] + \xi^\top(t)\dot{Q}(t)\xi(t) \\
&= \xi^\top(t)\{\dot{Q}(t) + \gamma[Q(t)F(t) + F^\top(t)Q(t) + \alpha^2 I + Q(t)F(t)F^\top(t)Q(t)]\}\xi(t) \\
&\quad - \frac{1}{\gamma}|\zeta(t) - \gamma F^\top(t)Q(t)\xi(t)|^2 - \frac{1}{\gamma}[\alpha^2\gamma^2|\xi(t)|^2 - |\zeta(t)|^2] \\
&\leq \xi^\top(t)\{\dot{Q}(t) + \gamma[Q(t)F(t) + F^\top(t)Q(t) + \alpha^2 I + Q(t)F(t)F^\top(t)Q(t)]\}\xi(t) \\
&= -\gamma\epsilon|\xi(t)|^2,
\end{aligned}$$

where we have used (26) to get the inequality. From the derivations above we conclude that the system (17) is exponentially stable. This stability is uniform with respect to $F(\cdot)$ since matrix P and numbers γ, ϵ do not depend on $F(\cdot)$. The proposition is proved. $\square\square\square$

Remark 5. Unfortunately, it is not clear at this point if the conditions of Proposition 5 are weaker or stronger than the ones of Proposition 3. Apparently, they are not related one to each other, and can only be compared on a case-by-case basis.

Remark 6. The proof of Proposition 5 was inspired by the work reported in [19].

5. Matrix F Depends on Time and the State

In this last section we consider the case when function F depends on t and x and $F(x, t)$ is inside a polytope with constant vertices, that is, there exists mappings $d_i : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i = 1, \dots, m$, such that

$$\begin{aligned}
F(x, t) &= \sum_{i=1}^m d_i(x, t)F_i, \quad \forall x, t \\
\sum_{i=1}^m d_i(x, t) &= 1.
\end{aligned} \tag{28}$$

5.1. An LMI condition for contraction

The variational system takes now the form

$$\dot{\xi}(t) = F(x(t), t)\nabla_x^2 H(x(t), t)\xi(t) + L(x(t), t)\xi(t), \tag{29}$$

where we defined the matrix

$$L(x, t) := [L_1(x, t) | \dots | L_n(x, t)], \quad L_j(x, t) := \left[\frac{\partial}{\partial x_j} F(x, t) \right] \nabla_x H(x, t), \quad j = 1, \dots, n. \tag{30}$$

Hence, the system (2) is contracting if the system (29) is UGES.

We assume that the functions $F(x, t)$ and $H(x, t)$ satisfy the following condition: there exists a non-negative, constant, $n \times n$ -matrix M such that

$$\frac{1}{2}[L(x, t) + L^T(x, t)] \leq M, \quad \forall x, t. \tag{31}$$

Notice that if $F(x, t)$ does not depend on x , then we can set $M = 0$.

Proposition 6. Consider the system (2) verifying (12) and the matrix $F(x(t), t)$ is Hurwitz for at least one t . Assume that (28) and (31) hold. If there exist a constant matrix $P = P^\top$ and positive numbers ϵ, τ such that the following LMIs hold

$$\begin{bmatrix} PF_j + F_j^\top P + \alpha^2 I + \epsilon I + \frac{\tau}{\gamma^2} M & PF_j & P \\ & F_j^\top P & -I & 0 \\ & P & 0 & -\tau I \end{bmatrix} \leq 0, \quad \forall j = 1, \dots, m, \quad (32)$$

with α defined in (14), then the system (2) is contracting.

Proof. Fix a solution $x(\cdot)$ of system (2). Rewrite equations (29) in the form

$$\dot{\xi} = \gamma F(x(t), t)\xi + F(x(t), t)\chi + \eta, \quad (33)$$

where we defined the signals

$$\chi(t) := [\nabla_x^2 H(x(t), t) - \gamma I] \xi(t), \quad \eta(t) := L(x(t), t)\xi(t).$$

Assume the following LMIs are feasible

$$\begin{bmatrix} QF_j\gamma + F_j^\top Q\gamma + \alpha^2\gamma^2 I + \epsilon I + \tau M & QF_j & Q \\ & F_j^\top Q & -I & 0 \\ & Q & 0 & -\tau I \end{bmatrix} \leq 0 \quad \forall j = 1, \dots, m. \quad (34)$$

Due to conditions (12) and (31) we have the following local quadratic constraints:

$$\begin{aligned} \alpha^2\gamma^2|\xi|^2 - |\chi|^2 &\geq 0 \\ \xi^\top M\xi - |\eta|^2 &\geq 0. \end{aligned}$$

Since for every $t \geq 0$ the matrix $F(x(t), t)$ belongs to a convex hull of matrices F_1, \dots, F_m , the inequality (34) implies

$$\begin{aligned} &2\xi^\top(t)Q[\gamma F(x(t), t)\xi(t) + F(x(t), t)\chi(t) + \eta(t)] \\ &+ \alpha^2\gamma^2|\xi(t)|^2 - |\chi(t)|^2 + \tau[\xi(t)M\xi(t) - |\eta(t)|^2] \leq -\epsilon|\xi(t)|^2. \end{aligned} \quad (35)$$

As in the proof of Proposition 3, set $V(\xi) = \xi^\top Q\xi$. Since the matrix $F(x(t), t)$ is Hurwitz at least for one t , matrix Q is positive definite. Using (35) we get that

$$\dot{V} \leq -\epsilon|\xi(t)|^2,$$

and, consequently, the system (29) is exponentially stable. Since Q, ϵ and τ do not depend on $x(\cdot)$, the stability is uniform with respect to $x(\cdot)$.

The proof is completed mimicking the calculations done in the proof of Proposition 3 to establish the equivalence of the LMI (34) with the LMI (32)—with a different ϵ and τ . $\square\square\square$

Remark 7. Clearly, the key assumption here is the bound (31), which seems hard to verify *a priori*.

5.2. An example of application of Proposition 6

Proposition 7. Consider the pH system (4) with

$$F(x, t) = \begin{pmatrix} -1 & \beta(x, t) \\ \beta(x, t) & -1 \end{pmatrix},$$

the Hamiltonian function $H(x, t)$ satisfying the inequality (12) and $\beta : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ verifying

$$|\beta(x, t)| \leq C_0, \quad |\nabla_x \beta(x, t)| |\nabla_x H(x, t)| \leq \frac{C'_0}{2}, \quad \forall x, t. \quad (36)$$

for some constants C_0, C'_0 . The system (4) is contracting provided

$$C_0^2 + \frac{2\sqrt{C'_0}}{c_1 + c_2} + \left(\frac{c_2 - c_1}{c_1 + c_2} \right)^2 < 1. \quad (37)$$

Proof. The proof is established verifying the conditions of Proposition 6. For, choose

$$F_1 := \begin{bmatrix} -1 & C_0 \\ C_0 & -1 \end{bmatrix}, \quad F_2 := \begin{bmatrix} -1 & -C_0 \\ -C_0 & -1 \end{bmatrix},$$

which satisfies (28) with

$$d_1(x, t) := \frac{1}{2} \left[1 + \frac{\beta(x, t)}{C_0} \right], \quad d_2(x, t) := \frac{1}{2} \left[1 - \frac{\beta(x, t)}{C_0} \right],$$

The columns of the matrix $L(x, t)$, defined in (30), are given by

$$L_j(x, t) = \begin{bmatrix} 0 & \frac{\partial \beta(x, t)}{\partial x_j} \\ \frac{\partial \beta(x, t)}{\partial x_j} & 0 \end{bmatrix} \nabla_x H(x, t).$$

Now, in view of (36), the entries of the vectors $L_j(x, t)$ are not bigger than $\frac{1}{2}C'_0$. Therefore,

$$\frac{1}{2}[L(x, t) + L^T(x, t)] \leq C'_0 I.$$

Hence, we can choose $M = C'_0 I$ to satisfy (31).

Let's set $P = I$. Then, the LMI (32) can be presented in the following equivalent form

$$F_j + F_j^\top + F_j F_j^\top + \alpha^2 I + \frac{\tau C'_0}{\gamma^2} I + \frac{1}{\tau} I < 0, \quad j = 1, 2. \quad (38)$$

Notice that

$$F_j + F_j^\top + F_j F_j^\top = (-1 + C_0^2) I, \quad j = 1, 2.$$

Replacing this identity in (38) yields

$$C_0^2 + \alpha^2 + \frac{\tau C'_0}{\gamma^2} + \frac{1}{\tau} \leq 1.$$

With $\tau = \frac{\gamma}{\sqrt{C_0}}$ the inequality takes the form

$$C_0^2 + \alpha^2 + \frac{\sqrt{C'_0}}{\gamma} < 1. \quad (39)$$

The proof is completed replacing the expressions of α and γ given in (14). □□□

Remark 8. The contraction condition (37) takes a particularly simple form when $H(x) = \frac{1}{2}|x|^2$, namely

$$C_0^2 + \sqrt{C'_0} < 1.$$

6. Conclusions

We have identified in the paper several classes of NLTV pH systems which are contracting. The contraction property is verified proving UGES of the variational system, which are shown in the paper to be equivalent properties. The case when the matrix F depends only on time or time and state, are treated separately. The classes are characterized by the solvability of—computationally efficient—LMIs, hence they can be reliably verified. For general pH systems, the LMI's are time/trajectory-dependent, but if the matrix F belongs to a convex polytope, then the LMIs are independent of time. The time dependence is also removed when F depends only on time, but in this case it is necessary to check a bi-linear matrix inequality.

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Appendix A

Theorem 1. [11, Theorem 1, Subsection 2.3] Suppose $f(t, x)$ is a \mathcal{C}^1 function in an open subset of \mathcal{D} of $\mathbb{R} \times \mathbb{R}^n$. For $(t_0, x_0) \in \mathcal{D}$ we denote by $\phi(t, t_0, x_0)$ the solution of the initial value problem

$$\dot{x} = f(t, x), \quad x(t_0) = x_0.$$

Then solution $\phi(t, t_0, x_0)$ is a \mathcal{C}^1 function. Moreover, the partial derivative of the solution with respect to the initial data x_0 , *i.e.*, $\frac{\partial \phi(t, t_0, x_0)}{\partial x_0}$, satisfies the initial value problem

$$\dot{\Phi}(t) = \frac{\partial f(t, x)}{\partial x} \Big|_{x=\phi(t, t_0, x_0)} \Phi(t), \quad \Phi(t_0) = I.$$

Appendix B

Lemma 1. Assume $n \times n$ time-varying matrices $A(t)$, $Q(t)$, $D(t)$, and constant $n \times n$ -matrices T , S , Γ , are such that for all $t \geq 0$ matrices $Q(t)$, $D(t)$, T , Γ are symmetric, $D(t) > 0$,

$$\begin{pmatrix} Q(t) & A(t) \\ A^\top(t) & D(t) \end{pmatrix} \leq \begin{pmatrix} T & S \\ S^\top & \Gamma \end{pmatrix}, \quad (.1)$$

and there exists a constant $n \times n$ -matrix $P = P^\top$ such that

$$\begin{pmatrix} PS + S^\top P + T & PS \\ S^\top P & -\Gamma \end{pmatrix} \leq 0. \quad (.2)$$

Assume c is a positive number, and $R(t)$ is a symmetric $n \times n$ matrix satisfying the following differential Riccati equation:

$$\dot{R}(t) = -c[R(t)A(t) + A^\top(t)R(t) + Q(t) + R(t)D(t)R(t)] \quad (.3)$$

with initial conditions such that $R(0) \geq P$.

Then, $R(t) \geq P$ for all $t \geq 0$.

Proof. Define the constant matrix

$$W := -PS - S^\top P - T - P\Gamma^{-1}P.$$

Then, $W \geq 0$ and $Z(t) \equiv P$ is a solution of the following differential Riccati equation

$$\dot{Z}(t) = -c[Z(t)S + S^\top Z(t) + T(t) + Z(t)\Gamma^{-1}Z(t) + W]. \quad (.4)$$

Since for all $t \geq 0$

$$\begin{pmatrix} Q(t) & A(t) \\ A^\top(t) & D(t) \end{pmatrix} \leq \begin{pmatrix} T+W & S \\ S^\top & \Gamma \end{pmatrix},$$

and $Z(0) \leq R(0)$. As shown in [[1], Theorem 4.1.4], we then have that $Z(t) \leq R(t)$ for all $t \geq 0$. The lemma is proved. $\square\square\square$

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