Interpolating Control with Periodic Invariant Sets

Sheila Scialanga, Sorin Olaru, Konstantinos Ampountolas

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***Abstract*— This paper presents a novel low-complexity interpolating control scheme involving *periodic invariance* or *vertex reachability of target sets* for the constrained control of LTI systems. Periodic invariance relaxes the strict one-step positively invariant set notion, by allowing the state trajectory to leave the set temporarily but return into the set in a finite number of steps. To reduce the complexity of the representation of the required controllable invariant set, a periodic invariant set is employed. This set should be defined within the controllable stabilising region, which is considered unknown during the design process. Since periodic invariant sets are not traditional invariant sets, a reachability problem can be solved off-line for each vertex of the outer set to provide an admissible control sequence that steers the system state back into the original target set after a finite number of steps. This work develops a periodic interpolating control (pIC) scheme between such periodic invariant sets and a maximal admissible inner set by means of an inexpensive linear programming problem, solved on-line at the beginning of each periodic control sequence. Proofs of recursive feasibility and asymptotic stability of the pIC are given. A numerical example demonstrates that pIC provides similar performance compared to more expensive optimization-based schemes previously proposed in the literature, though it employs a naive representation of the controllable invariant set.**

I. INTRODUCTION

Interpolating control (IC) has been proposed as a controller synthesis methodology for constrained dynamical systems in [1], [2]. The roots and the principles of IC can be found in the so-called vertex control, proposed in [3] for linear time-invariant discrete-time systems with polytopic state and control constraints, and later extended to uncertain plants in [4]. Vertex control builds on the existence of an admissible control action at each vertex of a controlled invariant set that pushes the state away from the boundary of the set as far as possible in a contractive sense. The vertex control uses the homogeneity of the dynamics and scale down the vertex control whenever the current state is in the interior or the original controlled invariant set. The limitation of this technique resides in the fact that the control action exploits the full control authority only on the border of the controllable invariant set and the convergence to the origin would be slower than a time-optimal control action [2]. To overcome this limitation, a switching control action has been proposed that applies a high gain stabilising state feedback controller when the state approaches the origin [5].

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IC emerges from the need to achieve a smooth transition from a low-gain/vertex-control that guarantees a large stabilizing set towards a high-gain feedback controller. The interpolation allows for the smooth transition between the two controllers and faster convergence to the origin of the state space [2]. The explicit version of the resulting controller has been characterized together with the geometrical properties [6] and extended to several interpolation factors and robustness [7]. A similar design philosophy was adopted in works dedicated to control sharing and merging [8] as well as in extensions to different classes of control Lyapunov functions [9], [10]. Recently, applications have been reported in automotive industry [11], transportation [12], and interconnected systems [13] .

The aforementioned IC schemes rely on the availability of "large" (ideally, maximal) controllable invariant sets. However, the approximation of the maximal controllable set is a tedious task both from the construction and from the representation point of view (complex half-space representation). The present paper aims to tackle this challenge by proposing a novel IC scheme that relaxes the one-step controlled invariance of the region which approximates the maximal controllable set. In place of the strict controlled invariant set within the interpolation scheme, the proposed periodic IC (pIC) employs a sequence of periodic sets starting from a pre-specified (and simple) initial set and given initial conditions. Periodic invariance guarantees that the system trajectories can be steered back in a finite number of steps by applying a sequence of apriori computed control actions. This notion can be resumed in terms of periodic invariance. Note that periodic invariant sets (PIS) have been used also in MPC to enlarge the stabilizing region and allow the state to leave the set and return after a finite number of steps [14], [15]. Similarly, pIC considers PIS to reduce the complexity of the representation of the invariant sets and avoid the computation of the expensive controllable invariant set.

This work also illustrates the simplicity of the representation of the feasible region that is obtained by solving a reachability problem on the vertices of the outer set for the constrained discrete-time system. Reachability of state-space regions or target sets for constrained discrete-time systems has been investigated in the past decades [16], [17], and it is currently a mature topic of research in control theory, thanks to advances in computational geometry [18], [19], [20]. Since for a particular outer set (e.g., rectangle or hyperbox) the vertices are known beforehand, the constrained reachability problem determines for each vertex of the outer set a sequence of admissible controls that steers the state of the system back into the original target set after a finite number of time steps. The computational complexity of pIC is concentrated in the off-line characterization of the reachable sets. For the interpolation, an inexpensive LP problem is developed and solved at the beginning of each periodic cycle. Proofs of recursive feasibility and stability of the proposed pIC scheme are given.

The rest of the paper is structured as follows. Section II introduces the problem under study and outlines some required definitions from invariant set theory. Section III presents the main results of this work including the proposed pIC scheme with constrained vertex reachability of target sets, and required proofs of recursive feasibility and stability. Section IV demonstrates the efficiency of pIC via a numerical example. Section V concludes the paper.

II. Preliminaries

A. System Dynamics and Constraints

Consider the discrete-time linear system,

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and control vectors, respectively; and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are known matrices. The state and control vectors of (1) are subject to polyhedral constraints:

$$\begin{cases} x(k) \in \mathcal{X},\, \mathcal{X} = \{x \in \mathbb{R}^n :\ F_x x \le g_x\}, \\ u(k) \in \mathcal{U},\, \mathcal{U} = \{u \in \mathbb{R}^m :\ F_u u \le g_u\}, \end{cases} \tag{2}$$

$\forall\, k \ge 0$, where $\mathcal{X}$ and $\mathcal{U}$ are described via half-space representation with F_x, F_u constant matrices and g_x, g_u constant vectors of appropriate dimension and with positive elements. The inequalities are considered component-wise and, consequently, the sets $\mathcal{X}$ and $\mathcal{U}$ are endowed with convexity and compactness properties and contain the origin as an interior point.

Assume that the pair (A, B) in (1) is controllable and thus a state-feedback controller $u(k) = Kx(k)$ exists, where $K \in \mathbb{R}^{m\times n}$ is a gain matrix. A state-feedback controller can be designed for unconstrained stabilisation with some user-desired performance specifications.

B. Set Invariance and Periodic Invariance

This section provides some definitions on the set invariance and periodic invariance [14], [21], [22], [23] that will be used in the rest of the paper.

Definition 2.1 (Constraint-admissible Invariant Set): The set $\Omega \subseteq \mathcal{X}$ is a positively constraint-admissible invariant set with respect to $x(k+1) = A_c x(k)$, where $A_c = A + BK$ is a closed-loop state matrix related to (1), subject to the local constraints (2), if $\forall\, x(k) \in \Omega$, the system evolution satisfies $x(k+1) \in \Omega$ and $Kx(k) \in \mathcal{U}$, $\forall\, k \ge 0$.

The largest positively invariant set for the system (1) in closed-loop with a static feedback control $u(k) = Kx(k)$ that respects constraints (2) is called *maximal admissible set* (MAS) [24]. Under stability and mild structural assumptions on the topology of the constraints (2) [3], [22], [18], MAS exists, it is finitely determined and can be defined in polyhedral form as,

$$\Omega = \{x \in \mathbb{R}^n : F_\Omega x \le g_\Omega\},$$

where F_Ω is a constant matrix and g_Ω is a constant vector of appropriate dimensions.

Definition 2.2 (Controllable Invariant Set): Given the system (1) and the constraints (2), the set $\Psi \subseteq \mathcal{X}$ is controllable invariant, if $\forall x(k) \in \Psi$, there exists an admissible control sequence $u(k) \in \mathcal{U}$ such that $x(k+1) \in \Psi$, $\forall\, k \ge 0$.

The maximal controllable invariant set Ψ might not be finitely determined within the class of polyhedral sets [21]. However, in the sequel, a polyhedral approximation will be considered with the half-space representation given by,

$$\Psi = \{x \in \mathbb{R}^n : F_\Psi x \le g_\Psi\}$$

where F_Ψ is a constant matrix and g_Ψ is a constant vector of appropriate dimensions.

For any scaling factor $\lambda > 0$, λS is understood as $\lambda S := \{\lambda x \,|\, x \in S\}$ for any set $S \subset \mathbb{R}^n$. Set invariance is a limit case of λ-contractiveness as indicated by the next definition.

Definition 2.3 (Controllable λ-contractive Set): Given a scalar $\lambda \in (0, 1]$, a set $\Psi \subseteq \mathcal{X}$ containing the origin is called controllable λ-contractive for (1) with respect to (2), if for any $x(0) \in \Psi$ there exists $u \in \mathcal{U}$ such that $x(k+1) \in \lambda\Psi$, for all $k > 0$.

For the special case of a contractive set with contraction factor $\lambda = 1$ is also called *controllable invariant*, see Definition 2.2. For a given λ, the maximal λ-contractive set, i.e., the union of all λ-contractive sets for (1) with respect to (2), is denoted by $\Psi^\lambda_{\max}$.

Definition 2.4 (Controllable Periodic Invariant Set [14]): For a given $\lambda \in \mathbb{R}_{[0,1)}$ the set $S \subset \mathbb{R}^n$ containing the origin is called *controllable periodic λ-contractive* with respect to the system (1) and constraints (2) if there exists a positive number $p \in \mathbb{Z}_+$ such that for any $x(k) \in S$ there exists an admissible control sequence $u(k+i) \in \mathcal{U}$, $i = 0, \ldots, p-1$, such that $x(k+p) \in \lambda S$ holds. If $\lambda = 1$ the set is called *controllable periodic invariant*.

Definition 2.5 (One-step Reachable Set): Given the system (1) with inputs, the set $\mathrm{Reach}(S)$ is called one-step reachable set from set S and contains the states that are reachable from S in one step with control action $u \in \mathcal{U}$,

$$\mathrm{Reach}(S) = \{x \in \mathbb{R}^n : \exists\, x(0) \in S, \exists\, u(0) \in \mathcal{U} \\ \text{s.t. } x = Ax(0) + Bu(0)\}.$$

C. Interpolating Control (IC) with Vertex Representation

IC relies on the (smooth) interpolation between a vertex controller and a conservative high-gain feedback controller. Fig. 1 depicts the idea behind the interpolating control technique. The set Ψ depicted in yellow is the *outer set*, e.g. controlled invariant set, and the MAS Ω is the *inner set* and it is depicted in red. The convex (polyhedral) outer and inner set are to be understood by the relationship $\Omega \subseteq \Psi \subseteq \mathcal{X}$. Any $x(k) \in \Psi$ can be decomposed as follows,

$$x(k) = s(k)\, x_v(k) + (1 - s(k))\; x_0(k), \tag{3}$$

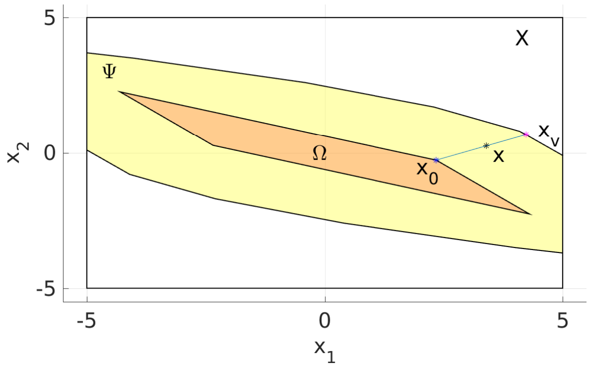

Fig. 1. The current state x can be decomposed as a convex combination of $x_v \in \partial\Psi$ and $x_0 \in \partial\Omega$.

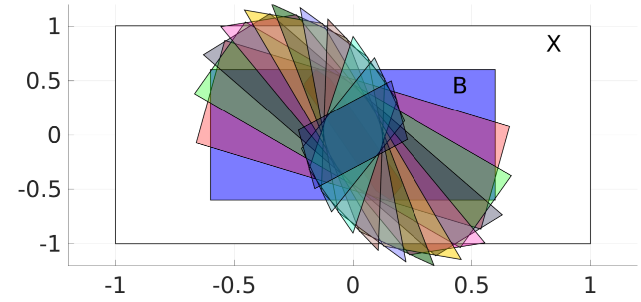

Fig. 2. Periodic invariant sets for the closed-loop equation $x(k+1) = (A + B\,K_1)\,x$, where $u = K_1\,x$ is the low-gain state feedback control input. The rectangle $\mathcal{B}$ is the starting set and the evolution of the state returns into the target set after 11 steps.

where $x_v(k) \in \Psi$ and $x_0(k) \in \Omega$, and $s(k) \in [0, 1]$ is the interpolating coefficient.

At each sampling instant, given the interpolation coefficient $s(k)$, one can obtain the control as follows,

$$u(k) = s(k)u_v(k) + (1 - s(k))u_0(k), \tag{4}$$

where $u_0(k) = K\,x_0(k)$ is an inner stabilising controller associated with the MAS and $u_v(k)$ is the vertex control applied to $x_v(k)$. The control (4) provides a smooth transition between the two controllers and a fast convergence to the origin of the state space.

Consider the change of variables $r_0 = (1-s)x_0$ and $r_v = sx_v$, where r_0, r_v are vectors of appropriate dimensions. It follows that $r_0 \in (1-s)\,\Omega$ and $r_v \in s\,\Psi$. The state decomposition (3) can be rewritten as $r_0 = x - r_v$. To solve the interpolation problem, an optimisation problem is formulated. The minimising problem is the following LP problem (index k is omitted for clarify):

$$\min_{s, r_v} s \qquad \text{subject to:} \begin{cases} sg_\Omega - F_\Omega r_v \leq g_\Omega - F_\Omega x, \\ -sg_\Psi + F_\Psi r_v \leq 0, \\ 0 \leq s \leq 1, \end{cases} \tag{5}$$

where the zero in the second inequality is a vector of zeros with length equal to the length of the vector g_Ψ. The solution of the LP problem is the interpolating coefficient s^* and the variable previously defined vector r_v^*. The original state variables can be recovered from $r_0^* = x - r_v^*$ with change of variables introduced previously. The solution of the optimization (5) leads to an admissible control action (4) at each time step that stabilises the constrained system [2]. Moreover, once the state enters the MAS – Ω, the interpolation control is equivalent to the stabilising high-gain feedback controller $u_0(k) = K\,x_0(k)$.

III. Periodic Interpolating Control

The IC presented in Section II-C relies on the availability of controllable invariant sets whereby the outer vertex controller is defined to enlarge the stabilising set. Moreover the complexity of vertex control might be high for high-order systems, which limits the applicability of the approach.

This section presents the main results of this paper. To overcome the complexity of the vertex control we employ *periodic invariant sets* as an ingredient for the interpolation procedure. The idea is to provide a simple alternative in case that controllable invariant sets cannot be determined or are unknown during the design process. It can also be seen as a possible enlargement of the stabilising region in the case when the state/input constraints can be relaxed to hold periodically.

A. Periodic Invariance

Consider a set with convenient representation (e.g. rectangle, hexagon). The set has to be defined in the controllable area of the constrained system (1)-(2) and needs to contain the MAS. Although the set is defined in the controllable area, it cannot guarantee its invariance with respect to the evolution of the state, since it is not an invariant set *per se*. In order to guarantee that the state will evolve towards the origin, we initially consider the constrained system (1)-(2) associated with a low-gain state-feedback controller $u(k) = K_1 x(k)$, where $K_1 \in \mathbb{R}^{m \times n}$ is a gain matrix, which asymptotically stabilises the system.

Fig. 2 shows an initial rectangle $\mathcal{B}$ that verifies the state constraints (in white) and the sequence of sets that starts from $\mathcal{B}$ and re-enters the target set in $p = 11$ steps. The sequence of sets is plotted to show the periodic invariance idea and how the period length is determined. Periodic invariant sets computed as reachable sets with low-gain controller (see Def. 2.5) do not verify the state constraints necessarily. Periodic invariance allows for the state vector to leave the invariant set temporarily but return into the set in a finite number of time steps, i.e., to leave the set for $k < p$, where p is the length of the period, and converge to the MAS at $k = p$.

Next section presents an interpolating control approach based on reachability enhancement. To this end, a constrained vertex reachability of target sets problem offers an admissible control sequence that steers the state of the system back into the original target set after a finite number of steps.

B. Periodic Invariance and Constrained Vertex Reachability

The proposed scheme with constrained vertex reachability of target sets involves *off-line* and *on-line* procedures. The *off-line* procedure involves an easy representation of the outer controllable invariant set (e.g., a rectangle or hexagon or octagon), and the solution of a *constrained reachability problem* for each vertex of the outer set. Since for a particular outer set (rectangle or hexagon or octagon) the vertices are known beforehand, the reachability problem determines for each vertex of the outer set a sequence of admissible controls that steer the state of the system back into the original target set after a finite number of time steps. The *on-line* procedure involves the interpolation between the MAS Ω and the simple outer set via the solution of an inexpensive LP problem.

1) Offline p-step Reachability Problem: Consider the linear time invariant system (1) subject to state and control constraints (2). Assume that a state feedback controller $u(k) = Kx(k)$ exists, which satisfies some user-desired performance specifications, and computes the maximal admissible set Ω associated to it. Ω plays the role of inner set in the proposed interpolating control scheme. Assume an outer set $\mathcal{B} \subseteq X$ with n parallel edges (e.g. in $\mathbb{R}^2$ for a rectangle $n = 2$, hexagon $n = 3$, octagon $n = 4$, etc.). Let v_i, $i = 1, \ldots, 2n$ be the vertices of the relevant outer set. The objective of the reachability problem is to compute a sequence of admissible controls u_{v_i} for each vertex v_i that steers the state of the system back into the target set $\mathcal{B}$ after a finite number of p_i steps. The constrained reachability problem allows us to satisfy the constraints, since the outer set representation (e.g. rectangle) is not an invariant set (see Fig. 2). In other words, it is not guaranteed that the state will remain inside the outer set at each time step without solving the constrained reachability problem. However, the periodic invariance property guarantees that the state trajectory will return into the target set at the end of the periodic sequence.

With a slight abuse of notation, denote $u_{v_i} = \{u_{v_i}(0), \ldots, u_{v_i}(p_i - 1)\}$ to be the p_i admissible control sequence for each vertex v_i, $i = 1, \ldots, 2n$. The controls u_{v_i} are obtained by the solution of the following constrained reachability problem for each vertex v_i:

$$\lambda^* \left(u_{v_i}(0), \ldots, u_{v_i}(p_i - 1)\right) = \min_{u_{v_i}, \lambda_i} \lambda_i$$

subject to:

$$\begin{cases} Av_i + Bu_{v_i}(0) \in \mathcal{X}, \\ A^2 v_i + ABu_{v_i}(0) + Bu_{v_i}(1) \in \mathcal{X}, \\ \vdots \\ A^{p_i-1} v_i + A^{p_i-2} Bu_{v_i}(0) + \cdots + Bu_{v_i}(p_i - 2) \in \mathcal{X} \\ A^{p_i} v_i + A^{p_i-1} Bu_{v_i}(0) + \cdots + Bu_{v_i}(p_i - 1) \in \lambda_i \mathcal{B} \\ u_{v_i}(k) \in \mathcal{U}, \quad k = 0, \ldots, p_i - 1, \\ 0 \leq \lambda_i < 1. \end{cases} \tag{6}$$

The solution of the reachability problem for each vertex v_i, $i = 1, \ldots, 2n$ of the target set $\mathcal{B}$ is a sequence of admissible controls that steers the vertex v_i into the target set in a contractive way, i.e., with a scaling factors λ_i (see Definition 2.3). The first p_i inequalites in (6) guarantee that the evolution of the state verifies the state constraints. The second to last inequalities guarantee that the controls $\{u_{v_i}(0), \ldots, u_{v_i}(p_i - 1)\}$ verify the control constraints, i.e. $u_{v_i}(k) \in \mathcal{U}$, $k = 0, \ldots, p_i - 1$. Finally, the last inequality steers the vertex v_i inside the target set $\mathcal{B}$.

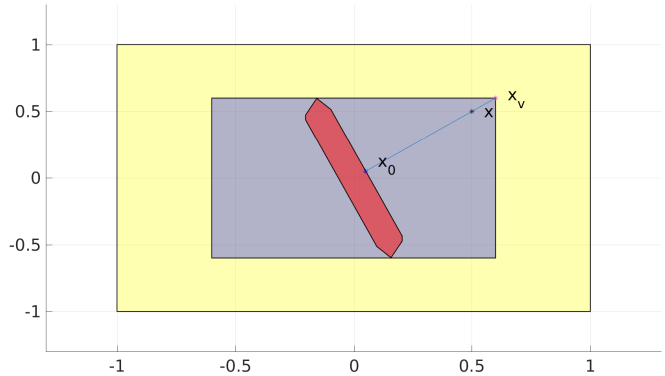

Fig. 3. Interpolating control concept with inner set Ω (red) and outer set the rectangle $\mathcal{B}$ (blue): $x_0 \in \partial\Omega$ and $x_v \in \partial\mathcal{B}$. The state constraints are the yellow rectangle.

The period length p_i, $i = 1, \ldots, 2n$, is defined for each vertex such that the reachability problem (6) has an admissible solution. A common period length for the constrained system (1)–(2) can be then defined as the least common multiple between all p_i, $i = 1, \ldots, 2n$:

$$p = \text{l.c.m. } p_i \quad i = 1, \ldots, 2n. \tag{7}$$

Any point x_v in the boundary of the outer set $\mathcal{B}$ can be defined as a convex combination of its vertices v_i. Then, there exists a sequence of p admissible control actions that steer the state of the system back into the target set in p steps. Note that the sequence of admissible controls $\{u_{v_i}(0), \ldots, u_{v_i}(p_i - 1)\}$, $i = 1, \ldots, 2n$ is stored in order to be accessed later on to steer the initial state $x \in \mathcal{B}$ back into the target set $\mathcal{B}$ via periodic interpolation.

2) Online pIC with Constant Interpolating Coefficient: Consider the initial state $x(0)$ that is defined inside the outer set $\mathcal{B}$ (and target set of periodic control). A scaling factor $\lambda_1^* \in [0, 1]$ can be computed such that the initial state is contained in the contractive rectangle $\lambda_1^* \mathcal{B}$. λ_1^* can be considered as the smallest contractive factor such that $x(0) \in \lambda_1^* \mathcal{B}$, and can be obtained by solving the LP problem:

$$\begin{aligned} &\lambda_1^* = \min_{\lambda} \lambda \\ &\text{subject to:} \\ &\begin{cases} F_{\mathcal{B}}\, x \leq \lambda\, g_{\mathcal{B}}, \\ 0 \leq \lambda \leq 1, \end{cases} \end{aligned} \tag{8}$$

where $F_{\mathcal{B}}$ and $g_{\mathcal{B}}$ are the matrix and the vector that defines the half-space representation of $\mathcal{B}$. Then, $\lambda_1^* \mathcal{B}$ can be set as the target set for our periodic control sequence.

The state $x(0)$ can be decomposed as $x(0) = s(0)\, x_v(0) + (1 - s(0))\, x_0(0)$ by solving the LP problem (5) with $\Psi = \mathcal{B}$. The states x_v and x_0 lie on the border of $\mathcal{B}$ and Ω, respectively (see Fig. 3). Then, $x_v(0)$ can be written as a convex combination of the vertices of the outer set $\mathcal{B}$, i.e.,

$$x_v(0) = \sum_{i=1}^{2n} \alpha_i(0)\, v_i, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{2n} \alpha_i = 1, \tag{9}$$

where α_i, $i = 1, \ldots, 2n$ are convexity coefficients in the unit simplex. The control action at $k = 0$ is a convex combination of the state feedback control applied to the state $x_0(0)$ and the combination of the controls applied to the vertices v_i, as in the decomposition (9), i.e.,

$$u(0) = s(0) \sum_{i=1}^{2n} \alpha_i(0)\, u_{v_i}(0) + (1 - s(0)) K x_0(0),$$

where $u_{v_i}(0)$ is the first element of the control sequence (6) applied to the vertex v_i. For the next $p-1$ steps, consider the p-sequence of interpolating controls, which are available from the reachability problem (6), to obtain the control,

$$u(k) = s(0) \sum_{i=1}^{2n} \alpha_i(0)\, u_{v_i}(k) + (1 - s(0)) K (A+BK)^k x_0(0), \tag{10}$$

for $k = 0, \ldots, p-1$. The control action (10) is applied to (1) for p steps or until the state reaches one of its target sets, i.e., either the contractive rectangle $\lambda_1 \mathcal{B}$ or the admissible set Ω. The control action (10) guarantees that the initial state $x(0)$ enters the contractive rectangle $\lambda_1 \mathcal{B}$ in p steps maximum. After the state returns into the rectangle, a new periodic sequence is computed. Note that in (10), the interpolating coefficient s and the coefficients α_i, $i = 1, \ldots, 2n$, in the convex combination (9) are kept constant, i.e. $s(k) = s(0)$ and $\alpha_i(k) = \alpha_i(0)$, $k = 1, \ldots, p$, $i = 1, \ldots, 2n$.

The contractive factor λ associated to the target set $\mathcal{B}$ is updated for the new state $x(\bar{k})$ by solving the LP problem (8), where $\bar{k}$ is the first time step of the periodic sequence. The current state would be inside $\lambda_2 \mathcal{B}$, $\lambda_2 < \lambda_1$, where $\mathcal{B}$ is the outer set of the periodic IC. After a new λ is obtained, a new interpolating decomposition $(s(\bar{k}), x_v(\bar{k}), x_0(\bar{k}))$ is computed between the outer set $\mathcal{B}$ and the inner set Ω with (5). The outer state is defined as convex combination of the vertices of the rectangle as in (9) with coefficients $\alpha_i(\bar{k})$, $i = 1, \ldots, 2n$. Similar to the control (10) applied to the initial state, a sequence of pIC associated to the new state is applied to the system, i.e.,

$$\begin{aligned} u(\bar{k}+k) = s(\bar{k}) \sum_{i=1}^{2n} \alpha_i(\bar{k}) u_{v_i}(\bar{k}+k) \\ + (1 - s(\bar{k})) K (A+BK)^k x_0(\bar{k}), \end{aligned} \tag{11}$$

for $k = 0, \ldots, p-1$, where $s(\bar{k})$ is the new interpolating coefficient to be kept constant in the new periodic sequence.

Algorithm 1 summarises the overall algorithmic scheme to determine the pIC using the constrained reachability problem outlined in this section. At the beginning of a periodic sequence, the contractive factor λ is computed and a state decomposition (3) is obtained as solution of the optimisation problem (5). The periodic interpolating control action (11) is then applied to the state for p steps (see (7)) or until it reaches one of its target sets, i.e., either $\lambda_j \mathcal{B}$, where j is the prevailing periodic sequence, or the admissible set Ω paired to a stabilising feedback controller. If the state enters the scaled target set, a new periodic sequence starts afterwards. On the other side, if the state enters the MAS in less than p steps, the control action becomes the state-feedback controller $u = Kx$ (i.e., $s = 0$).

Algorithm 1: pIC: Periodic interpolating control using constrained vertex reachability

```
input : System matrices A, B; High-gain feedback
        matrix K; Sets X, U; Outer/target set B
        (rectangle or hexagon or octagon); Number
        of steps N.
output: State evolution x; Applied control u;
        Interpolating coefficient s.
Solve the reachability problem (6) for each vertex v_i,
  i = 1,...,2n:
  • Store the control actions u_{v_i};
  • Determine a common period p for the overall system
    as l.c.m.(p_{v_i});
Define the initial state x(0) that belongs to B;
Define x ← x(0), l ← 0, k̄ ← 0, λ_x ← 1;
for i = 1 to N do
    if x ∉ Ω then
        if (l = p) or (x ∈ λ_x B) then
            Compute the scaling factor λ_x = min λ,
            λ ∈ [0,1], such that x ∈ λ_x B;
            Compute (s, x_v, x_0) solving the LP
            problem (5);
            Set k̄ ← k̄ + l;
            Apply the control u (11) with k = 0;
            Set l ← 1;
        end
        else
            Apply the control u (11) with k = l;
            Set l ← l + 1;
        end
    end
    else
        Apply the control u = Kx;
        Set s ← 0;
    end
    Update x = Ax + Bu
end
```

Remark 3.1: Periodic interpolating control is introduced in order to provide an easy half-space representation of the controllable invariant set and reduce the computational complexity of the online conmputations. Section III-B.2 assumes polyhedron with parallel edges. However, this approach can be applied to any convex polytope. One request might be that contains a set of initial points of interest.

C. Recursive Feasibility and Asymptotic Stability

This section provides the necessary proofs of recursive feasibility and asymptotic stability of the proposed periodic interpolating control in Section III-B.1 and III-B.2 with constrained vertex reachability for the linear system (1)-(2).

1) p-step feasibility: For p-step feasibility of the pIC, we need to prove that $u(k) \in \mathcal{U}$ and that if the state $x(k)$ is feasible at time k, it will be also feasible at time $k+p$. In other words there exists an admissible control sequence $u(k) \in \mathcal{U}$ that steers the state in the feasible set in p steps. Let u_{v_i} be the vector of admissible control sequence $\{u_{v_i}(0), \dots, u_{v_i}(p-1)\}$ that steers each vertex v_i, $i = 1, \dots, 2n$ into the rectangle as solution of the reachability problem (6), and let p be the number of time steps required to bring the states contained in the outer set $\mathcal{B}$ back into the target set. The next theorem provides a proof of the p-step reachability problem presented in Section III-B.

Theorem 3.1: The periodic interpolating control (3), (4), (5), (11) is p-step feasible for the linear time invariant system (1) with state and control constraints (2) and for all states inside the feasible region $\mathcal{B}$. That is, the state will return inside the feasible set after p steps, i.e.,

$$\forall x(k) \in \mathcal{B} \implies x(k+p) \in \mathcal{B}, \quad k \geq 0.$$

Proof: We want to prove that $u(k) \in \mathcal{U}$ for all $k \geq 0$. The control actions needs to verify $F_u u(k) \leq g_u$, with $u(k) = s(k)\, u_v(k) + (1 - s(k))\; u_0(k)$, $\forall k \geq 0$. Firstly, we prove that the outer control u_v verifies the control constraints (the index k is omitted for clarity):

$$u_v = \sum_{i=1}^{2n} \alpha_i\, u_{v_i}, \qquad \alpha_i \geq 0, \qquad \sum_{i=1}^{2n} \alpha_i = 1 \tag{12}$$

$$\begin{aligned} F_u\, u_v &= F_u \sum_{i=1}^{2n} \alpha_i\, u_{v_i} \\ &= \sum_{i=1}^{2n} \alpha_i F_u\, u_{v_i} \leq \sum_{i=1}^{2n} \alpha_i\, g_u = g_u. \end{aligned} \tag{13}$$

The last inequality in (13) holds because u_{v_i} is one of the control actions and solutions of the reachability problem (6). We now prove that the control action (11) is admissible:

$$\begin{aligned} F_u u(k) &= F_u \left(s(k)\, u_v(k) + (1 - s(k))\; u_0(k) \right) \\ &= s(k) F_u u_v(k) + (1 - s(k)) F_u u_0(k) \\ &\leq s(k) g_u + (1 - s(k))\, g_u \\ &= g_u, \end{aligned} \tag{14}$$

where the last inequality hold from (13), where u_0 is control action within the MAS Ω.

Now we go back to prove that the set $\mathcal{B}$ is p-step feasible set, that is, for all $x(\bar{k}) \in \mathcal{B}$, then $x(\bar{k}+p) \in \mathcal{B}$. Consider the state decomposition,

$$x(\bar{k}) = s(\bar{k})\, x_v(\bar{k}) + \left(1 - s(\bar{k})\right)\; x_0(\bar{k}),$$

obtained from the solution of the LP problem (5). The inner state $x_0 \in \Omega$ evolves with the stabilising high-gain feedback controller as the closed-loop system $x_0(\bar{k}+1) = (A + B\, K)\, x_0(\bar{k})$, while the outer state $x_v \in \mathcal{B}$ is contained in the boundary of the set $\mathcal{B}$ and evolves with the outer controller u_v. During the p-step periodic sequence, x_v evolves according to: $x_v(\bar{k}+k) = A\, x_v(\bar{k}+k-1) + B\, u_v(\bar{k}+k-1)$ for $k = 1, \dots, p$. After p steps, the state $x_v(k)$ returns into the target set $\mathcal{B}$ because of the construction of the periodic invariant set sequence, i.e., $x_v(\bar{k}+p) \in \mathcal{B}$. The inner state evolves according to: $x_0(\bar{k}+k) = (A + B\, K)^k\, x_0(k)$ for $k = 1, \dots, p$. Since Ω is computed as the MAS of the system with control $u = Kx$, after p steps the state $x_0(\bar{k})$ is inside the MAS, i.e. $x_0(\bar{k}+p) \in \Omega$.

From the state decomposition (3) and the interpolating coefficient $s(\bar{k})$ computed for $x(\bar{k})$, we obtain that the outer state evolves according to:

$$\begin{cases} s(\bar{k}) x_v(\bar{k}+k) = s(\bar{k})(A\, x_v(\bar{k}+k-1) B\, u_v(\bar{k}+k-1)) \\ s(\bar{k}) x_v(k+p) \in s(\bar{k})\mathcal{B}, \end{cases} \tag{15}$$

for $k = 1, \dots, p-1$, while the inner state $x_0(k)$ evolves according to:

$$\begin{cases} (1 - s(\bar{k})) x_0(\bar{k}+k) = (1 - s(\bar{k}))\, (A + B\, K)^k\; x_0(\bar{k}), \\ \left(1 - s(\bar{k})\right) x_0(\bar{k}+p) \in \Omega. \end{cases} \tag{16}$$

for $k = 1, \dots, p-1$. The initial state $x(\bar{k})$ after p steps is decomposed as,

$$x(\bar{k}+p) = s(\bar{k}) x_v(\bar{k}+p) + (1 - s(\bar{k})) x_0(\bar{k}+p),$$

where $s(\bar{k}) x_v(\bar{k}+p) \in s(\bar{k})\mathcal{B}$ and $(1 - s(\bar{k})) x_0(\bar{k}+p) \in \Omega$. Since $\Omega \subseteq \mathcal{B}$, it follows that $(1 - s(\bar{k}))\Omega \subseteq (1 - s(\bar{k}))\mathcal{B}$ and $x(\bar{k}+p)$ is convex combination of points in $\mathcal{B}$, i.e. $x(\bar{k}+p) \in \mathcal{B}$, which concludes the proof. ■

2) Asymptotic stability: The following theorem provides a proof of asymptotic stability.

Theorem 3.2: The periodic interpolating control (3), (4), (5), (11) guarantees asymptotic stability of the linear time invariant system (1) with state and control constraints (2) for any initial point $x(0) \in \mathcal{B}$. *Proof:* We want to prove that for each initial state in the feasible set, x converges to Ω in finite time. Consider $V(x(k)) = s^*(x(k))$, $\forall x \in \mathcal{B} \setminus \Omega$ as candidate Lyapunov function. After solving the LP problem (5) for the state $x(k)$ and applying the control (11) for p steps, one obtains,

$$x(k+p) = s^*(x(k))\, x_v(k+p) + (1 - s^*(x(k)))\, x_0(k+p),$$

where $x_v(k+p) \in \mathcal{B}$ is the outer state obtained after applying the outer controller for p steps and $x_0(k+p) = (A + B\, K)^p\, x_0(k) \in \Omega$. Hence, $s^*(x(k))$ is feasible solution of the LP problem (5) at time $k+p$. Solving the LP (5) again, one gets $s^*(x(k+p))$ and state decomposition $x(k+p) = s^*(x(k+p))\, x_v^*(k+p) + (1 - s^*(x(k+p)))\, x_0^*(k+p)$ with $x_v^*(k+p) \in \mathcal{B}$ and $x_0^*(k+p) \in \Omega$. Since $s^*(x(k+p)) \leq s^*(x(k))$, the candidate Lyapunov function $V(x)$ is non-increasing. Furthermore, since the outer controller u_v is contractive, it guarantees convergence to Ω in finite time. Inside Ω, the interpolating coefficient $s(x)$ is null and the control action is the stabilising state feedback control $u = Kx$. Finally, since the local feedback controller is contractive, asymptotic stability is guaranteed for all $x \in \mathcal{B}$ with control action (11). ■

IV. Numerical Example

Consider the discrete-time linear system with two state and one control variables in [25]. The state and control matrices are as follows

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}. \tag{17}$$

State and control variables are subject to constraints,

$$|x_i| \leq 1 \text{ with } x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, i = 1, 2 \text{ and } |u| \leq 2. \tag{18}$$

Our goal is to compute with the proposed periodic interpolating control (pIC) using constrained vertex reachability of target sets presented in Section III a control action that satisfy the state and control constraints (18) at each time step.

System (17) is controllable and a state feedback controller that stabilise the system can be defined as $u = K\,x$ with,

$$K = -\begin{bmatrix} 30.3781 & 9.6139 \end{bmatrix}. \tag{19}$$

The closed-loop system $A + B\,K$ has complex eigenvalues $\lambda_{1,2} = 0.6167 \pm 0.3036\imath$ with module $0.6874 < 1$. Given that the module of the eigenvalues is smaller than the unit, the system is stable and Ω is the maximal admissible set that verifies the system constraints (18) with a high-gain state feedback matrix (19). Fig. 4(e) depicts in red the set Ω with half-space representation $\Omega = \left\{x \in \mathbb{R}^2 \colon F_\Omega\, x \leq g_\Omega\right\}$.

In order to apply the pIC strategy, consider a rectangle with edges parallel to the axis $\mathcal{B} = \{x \colon F_\mathcal{B} x \leq g_\mathcal{B}\}$ with,

$$F_\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad g_\mathcal{B} = \begin{bmatrix} 0.6 \\ 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}, \tag{20}$$

that contains the maximal admissible set Ω. $\mathcal{B}$ is depicted in blue colour in Fig. 4(e) and plays the role of outer set in the computation of periodic interpolating control. Note that the choice of the rectangle is arbitrary but it has to verify the state constraints and has to contain the inner stabilisable set Ω, i.e. $\Omega \subseteq \mathcal{B} \subseteq \mathcal{X}$. The maximal controllable set $\Psi = \left\{x \in \mathbb{R}^2 \colon F_\Psi\, x \leq g_\Psi\right\}$ is depicted in yellow and will be used to implement the IC approach in [26]. Note that Ψ is considered unknown for the proposed pIC in Section III.

The set $\mathcal{B}$ is not controllable invariant. It is chosen as a subset of the maximal controllable set Ψ, and thus for every state $x \in \mathcal{B}$ a control action can be computed that verifies the control constraints $u \in \mathcal{U}$ while the evolution of the state may not be within the rectangle $\mathcal{B}$. However, convergence into the target rectangle is guaranteed in a finite number of steps with a sequence of admissible controls that can be obtained using the reachability problem (6).

As first step in order to implement the pIC for the constrained system (17)–(18), four reachability problems (6) for the four vertices of the rectangle $\mathcal{B}$ are solved. The four vertices of $\mathcal{B}$ read: $v_1 = [0.6\ 0.6]^\mathsf{T}$, $v_2 = [-0.6\ 0.6]^\mathsf{T}$, $v_3 = [-0.6\ -0.6]^\mathsf{T}$, and $v_4 = [0.6\ -0.6]^\mathsf{T}$. From the solution of the reachability problem (6) for the four vertices v_i, $i = 1, \ldots, 4$, we obtain the periodicity of each vertex with $p_1 = p_3 = 9$ steps and $p_2 = p_4 = 1$ steps. Thus the least common period length equals to $p = 9$ of the LTI system.

Figs 4(a)–4(c) show the state and control trajectories for the initial state $x(0) = [0.55\ 0.55]^\mathsf{T}$ under pIC and traditional IC [26] (where Ψ is used). As can be seen, both approaches exhibit similar state trajectories, albeit with different control actions and different interpolating coefficients (see Fig. 4(d)). The scaling factors λ are $\lambda_{pIC} = \{0.9167, \quad 0.8560\}$. The control effort of the two approaches is: $\|u_{\mathrm{pIC}}\|_2 = 7.1749$ and $\|u_{\mathrm{IC}}\|_2 = 6.3172$ for pIC and IC, respectively. Obviously the pIC needs more effort to stabilise the system due to periodic invariance but is less computational expensive and employs as outer set Ψ a simple rectangular representation.

Fig. 4(d) depicts the interpolating coefficients for the two methods. As can be seen, the interpolating coefficient of pIC takes the value $s = 0.9$ and remains constant over $p = 9$ steps. Then, it decreases to $s = 0.8$ for the new periodic sequence with $p = 5$ steps. Finally, it takes the value $s(15) = 0$ at $k = 15$ because the system state has entered the MAS, and thus the high-gain state-feedback controller is applied. On the other hand, the interpolating coefficient for IC is decreasing at every time step until reaches $s = 0$, that is, the state x is in the MAS. In both approaches, the interpolating coefficient plays the role of Lyapunov function that guarantees convergence. It should be noted that the two approaches converge to the MAS at $k = 15$, see Fig. 4(d) (i.e., their interpolating coefficients $s(15) = 0$ at $k = 15$). Fig. 4(f) shows the evolution of system state in the $\mathbb{R}^2$-space under the two methods for the initial point $x(0) = [0.55 \quad 0.55]^\mathsf{T}$.

V. CONCLUSIONS

This work presented a novel low-complexity interpolating control scheme with periodic invariance and constrained vertex reachability of target sets for linear systems with state and control constraints. It relies on an easy representation of the outer controllable invariant set (e.g., a rectangle or hexagon or octagon), and then solves a reachability problem for each vertex of the outer set for the constrained discrete-time system under study. Since for a particular outer set (e.g. rectangle) the vertices are known beforehand, the constrained reachability problem determines for each vertex of the outer set a sequence of admissible controls that steer the state of the system back into the original target set after a finite number of time steps. For the interpolation, an inexpensive LP problem is solved at the beginning of each periodic cycle.

Proofs of p-step recursive feasibility and asymptotic stability of the periodic IC scheme are given. The numerical example demonstrated that the proposed approach, although employed a naive rectangular representation of the controllable invariant set Ψ, provided similar performance to the more expensive traditional IC while it guaranteed convergence and satisfaction of the state and control constraints. Results in this work were demonstrated in the $\mathbb{R}^2$-space, though their extension to high-dimensional spaces is straightforward (e.g., a box or cube can be used in $\mathbb{R}^3$-space instead of a rectangle).

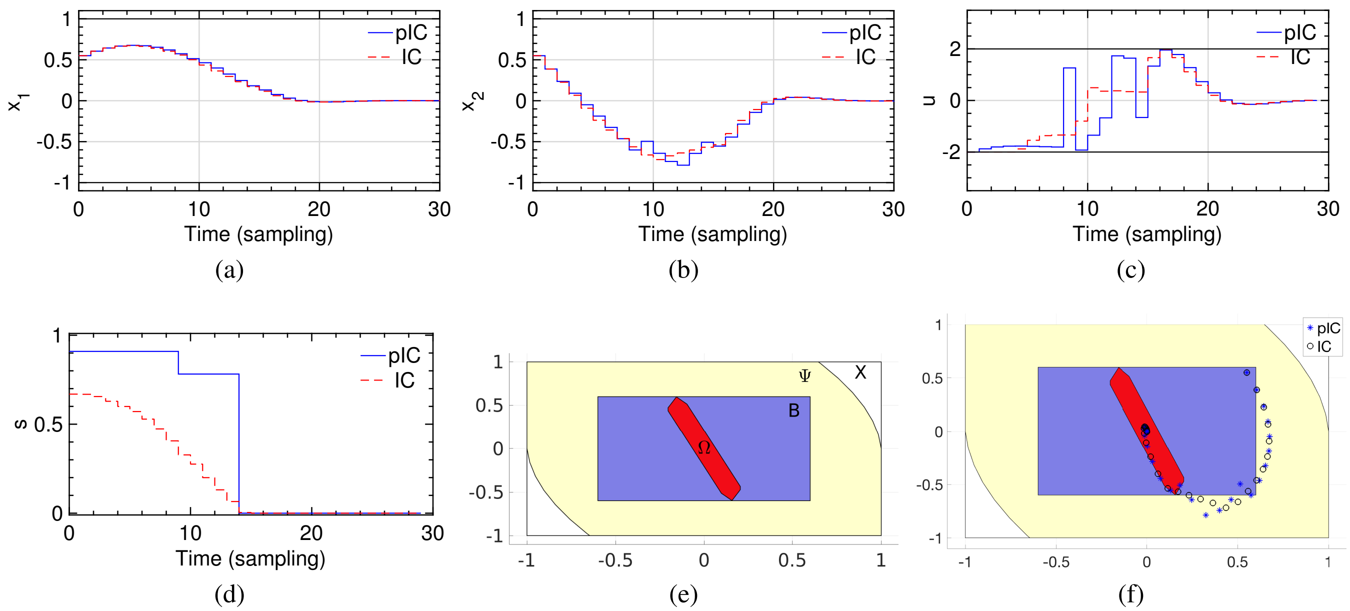

Fig. 4. (a), (b), (c): state and control trajectories for the initial state $[0.55, 0.55]^{\mathsf{T}}$; (d): interpolating coefficient; (e) Invariant sets; (f) State evolution for the initial state $[0.55, 0.55]^{\mathsf{T}}$. pIC is marked with blue stars and standard IC with black circles.

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