Quantum Inspired Eigenlogic and Truth Table Semantics

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Zeno Toffano
Laboratoire des signaux et systmes - CNRS
Gif-sur-Yvette, France
CentraleSupélec
Université Paris-Saclay
Gif-sur-Yvette, France
zeno.toffano@centralesupelec.fr

Eigenlogic proposes a new method in logic inspired from quantum theory using operators. It expresses logical propositions using linear algebra. Logical functions are represented by operators and logical truth tables correspond to the eigenvalue structure. It extends the possibilities of classical logic by changing the semantics from the Boolean binary alphabet $\{0,1\}$ using projection operators to the binary alphabet $\{+1,-1\}$ employing reversible involution operators. Also, many-valued logical operators are synthesized, for whatever alphabet, using operator methods based on Lagrange interpolation and on the Cayley-Hamilton theorem. Considering a superposition of logical input states one gets a fuzzy logic representation where the fuzzy membership function is the quantum probability given by the Born rule. Eigenlogic is essentially a logic of operators and its truth-table logical semantics is provided by the eigenvalue structure which is shown to be related to the universality of logical quantum gates, a fundamental role being played by non-commutativity and entanglement.

1 Eigenlogic

In this extended abstract a general overview of a new approach in logic named Eigenlogic is presented. The technical details are in the referenced papers, the most recent one [7] gives the latest developments. Eigenlogic is an operational and geometric approach to logic. It stems from the multilinear elective decomposition of binary logical functions in the original form introduced by George Boole. A justification on historical grounds is presented in [3] bridging Boole's elective decomposition theory with the axioms of Boolean algebra using sets and projection operators. It is shown that Boole's arithmetic polynomial formulation can be naturally extended to operators in vector spaces. In this way propositional logic can be formalized in linear algebra by using tensor product combinations of elementary operators.

The interesting feature is that the eigenvalues of these operators become the truth values of the corresponding logical connectives and the associated eigenvectors correspond to one of the fixed combinations of the logical inputs (logical interpretations), showing that truth tables can be derived in an unique way for every logical operator. The outcome of a measurement on a logical operator is the truth value of the associated logical proposition, and becomes logically interpretable when applied to the operator's eigenspace, leading to a natural analogy with the measurement postulate in quantum mechanics.

The following diagram summarizes this point of view:

$$\begin{aligned} \text{eigenvalues} &\longleftrightarrow \text{truth values} \\ \text{eigenvectors} &\longleftrightarrow \text{interpretations} \\ \text{logical operators} &\longleftrightarrow \text{logical connectives} \end{aligned}$$

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The definition of *atomic propositions* in Eigenlogic is different from the one in traditional quantum logic [3]. In propositional logic atomic propositions must be independent propositions and correspond to the so-called *logical projectors* (named also *dictators* when using numbers different from the booleans $\{0,1\}$). In Eigenlogic propositions are represented by operators and independence can only be achieved by an extension with the identity operator $\boldsymbol{I}$ by the Kronecker product. For example, for arity-2, the two logical projectors are $\boldsymbol{A} = \Pi \otimes \boldsymbol{I}$ and $\boldsymbol{B} = \boldsymbol{I} \otimes \Pi$, which are rank-2 projection operators, where the operator $\Pi = |1\rangle\langle 1|$ represents the one-qubit density matrix (a rank-1 projection operator) and is in Eigenlogic the *seed operator*. This is a fundamental difference with quantum logic where atomic propositions are associated with rays (rank-1 projection operators) corresponding to pure quantum state density matrices. In Eigenlogic a ray corresponds to logical conjunction (*AND*, $\wedge$), given, for arity-2, by the operator $\boldsymbol{A} \cdot \boldsymbol{B} = \Pi \otimes \Pi$ which is known to be a non-atomic connective [3].

One can generalize Eigenlogic to truth-values different from the Boolean binary values $\{0,1\}$, for example using $\{+1,-1\}$ associated to self-inverse involution unitary operators [1, 7]. In general one can associate a binary logical operator with whatever couple of distinct eigenvalues $\{\lambda_1, \lambda_2\}$, the corresponding family of logical operators can be found by Lagrange-Cayley-Hamilton matrix interpolation methods.

The extension from binary to many-valued logic is then straightforward by defining specific operators using multivariate interpolation [1, 5, 7]. The interesting property is that a unique seed operator generates the complete logical family of Eigenlogic operators for a given m-valued n-arity system. It has to be outlined that in a complete logical family all Eigenlogic operators commute.

Considering non-eigenvectors as inputs, the logical operators are no more diagonal and can be understood as *fuzzy logic* propositions [1, 7]: the degree of truth, being no more a sharp truth value, becomes the fuzzy membership function given by the quantum mean value (Born rule) of the Eigenlogic operator on the input state. When considering Eignelogic projection operators with eigenvalues $\{0,1\}$ the mean value gives the quantum probability.

2 Applications of Eigenlogic

The original and principal motivation of this research is for applications in the field of quantum computing and quantum information. The correspondence of Eigenlogic with standard quantum computing gates is thoroughly discussed in [5]. The common control gates can be obtained by considering the correspondence between the control logic gates and binary Eigenlogic involution operators. A new polynomial and exponential derivation of the TOFFOLI-gate is presented, the method has parallels to standard quantum $\boldsymbol{T}$-gate optimization methods.

There is an increasing interest in applying many-valued logic to the interpretation of quantum phenomena. Many-level systems (having a state-space dimension greater than two) are ubiquitous in quantum physics, an important example is quantum angular momentum where the binary spin $\frac{1}{2}$ system is a particular case. Ternary-logic quantum gates using qutrits lead to less complex circuits, the design of a balanced qutrit arithmetic full-calculator circuit has been prposed using an Eigenlogic approach in [5].

Fuzzy Eigenlogic was applied to the concept of quantum robot introduced by Paul Benioff as a first approach for describing a quantum mechanical system aware of the environment and capable of making decisions. The quantum robot models were designed by the means of Braitenberg vehicles with Eigenlogic control and fuzzy input vision stimuli in [2] and [6]. These robots display new non-classical emergent behaviors linked to quantum-like effects and reflect contextuality due to input-state superposition and to the entanglement of the logical control structure.

3 Perspectives for Eigenlogic

Boole's logical interpretation of quaternions is considered in [7]. The isomorphism between quaternions and Pauli matrices leads to a natural quantum interpretation in Eigenlogic showing that the logical operation of binary qubit negation is a consequence of the Pauli matrix anti-commutativity.

The generalized Pauli operators, the so-called *Weyl-pairs* $\boldsymbol{Z}_d$ and $\boldsymbol{X}_d$ for a d-dimensional system are considered as Eigenlogic operators. Their many-valued truth table semantics is given by the eigenvalue structure of the seed operator $\boldsymbol{Z}_d$ in the computational basis, where the d eigenvalues are the roots of unity ${\omega_d}^k = e^{\frac{2i\pi k}{d}}$ with $k \in \{0,\ldots,d-1\}$. In a binary system this corresponds to the square roots of unity, ± 1, the two eigenvalues of the diagonal $\boldsymbol{Z}$-gate (PHASE-gate). The syntax on the other hand is represented by the shift operator $\boldsymbol{X}_d$ which transforms computational basis states as $\boldsymbol{X}_d\,|k\rangle = |k+1\rangle$, where k is a modulo-d number. In a binary system this operator is the $\boldsymbol{X}$-gate (NOT-gate).

Logical truth-table semantics is naturally represented by the eigenstructure of these Eigenlogic operators. Interestingly the Quantum Fourier Transform, the unitary transformation between $\boldsymbol{Z}_d$ and $\boldsymbol{X}_d$ (the HADAMARD-gate for a binary system), can be viewed in Eigenlogic as the logical semantics-syntax mediator. The operators being reversible, the same argument is valid for the inverse transformations.

Linear algebra has become the standard tool of computer based disciplines related to Big Data. Formalizing logic in an operator matrix language, as is done in Eigenlogic, can bring benefits because logic can be treated directly in this framework. Inspired by several approaches in logic, conjectures are proposed using quantum gates and quantum algorithms in an Eigenlogic framework:

- *quantum-gate universality* related to the semantic truth-table Eigenlogic structure [4, 7].
- example of an Eigenlogic *first order proposition* seen as a *Grover search* gate [4, 7].
- non-commutative operator formulation of *Post normal systems* [7].
- reversible quantum gate formulation for *combinatory logic* [7].

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