# Quantum Entanglement and the Lorentz Group

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The Lorentz metric represented by the diagonal matrix $\boldsymbol{G} = \mathrm{diag}(1, -1, -1, -1)$ acts on Minkowski space-time quadrivectors. In the language of Quantum Information the operator $\boldsymbol{G}$ can be viewed as an entangling gate this because it acts in a similar way as the Controlled-Z gate on the computational basis of a 2-qubit separable quantum vector. The entangling power corresponds to the fact that the resulting vector, considered as a 2-qubit vector, cannot be put into a Kronecker product of two 1-qubit vectors. For example considering a uniform positive normalized input vector, which is separable, one has the transformation :

$$\boldsymbol{G} \cdot \tfrac{1}{2}(1, 1, 1, 1)^T = \tfrac{1}{2}(1, -1, -1, -1)^T$$

It can be easily verified that in this case the resulting output vector is completely entangled, for example by calculating the associated quantum concurrence which equals to 1.

One can represent the generators of the Lorentz group by $4 \times 4$ matrices. An example is given by the Lorentz rotation matrix $\boldsymbol{Z}(\phi)$ [1] :

$$\boldsymbol{Z}(\phi) = \exp(-i\phi \boldsymbol{J}_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} , \; \boldsymbol{J}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \boldsymbol{\Pi}_1 \otimes \boldsymbol{\sigma}_y$$

The matrix $\boldsymbol{J}_3$ is the associated Lorentz group generator. $\boldsymbol{J}_3$ can be expressed as the Kronecker product of the qubit logical-1 projector $\boldsymbol{\Pi}_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ with the Pauli matrix $\boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and consequently the operator $\boldsymbol{Z}(\phi)$ in a qubit representation acts as a control gate on the unitary double angle rotation operator $\boldsymbol{R}_y(2\phi) = \exp(\text{-}i\phi\boldsymbol{\sigma}_y)$ around the $Oy$ axis. This can be highlighted by developing the exponential and using the idempotent property of the projector and the involution property of the Pauli matrix :

$$\boldsymbol{Z}(\phi) = \exp(-i\phi(\boldsymbol{\Pi}_1 \otimes \boldsymbol{\sigma}_y)) = \boldsymbol{\Pi}_0 \otimes \boldsymbol{I}_2 + \boldsymbol{\Pi}_1 \otimes \boldsymbol{R}_y(2\phi)$$

where $\boldsymbol{\Pi}_0 = \boldsymbol{I}_2 - \boldsymbol{\Pi}_1 = |0\rangle\langle 0|$ is the qubit logical-0 projector. The form can be compared with the more-known entangling gate Control-NOT which can be expressed also as a function of projectors :

$$\boldsymbol{C}_{NOT} = \boldsymbol{\Pi}_0 \otimes \boldsymbol{I}_2 + \boldsymbol{\Pi}_1 \otimes \boldsymbol{\sigma}_x = \boldsymbol{\Pi}_0 \otimes \boldsymbol{I}_2 + \boldsymbol{\Pi}_1 \otimes i\boldsymbol{R}_x(\pi)$$

which is a control gate on the unitary Pauli operator $\boldsymbol{\sigma}_x = \boldsymbol{X}$ (the NOT gate) and corresponds geometrically to a reflection or equivalently a rotation around the $Ox$ axis of angle $\pi$ multiplied by the imaginary number $i$.

The following questions arise : is there a link between the Lorentz space-time structure and entanglement ? Do they have a common origin ? These questions could be related with the lack of mechanical understanding of the nature of a relativistic spinor. Spinors are often identified with qubits, for example the complex 1-qubit column vector $|\psi\rangle = (\psi_1, \psi_2)^T$ transforms under left-multiplication with matrices in the $SU(2)$ special-unitary group like a 2-spinor [2] and Dirac 4-spinors can be associated with 2-qubit states. The physics related to the information content of the Lorentz group is often overlooked, not being considered for actual applications in Quantum Information. Spinor algebra arises by the Lorentz invariance constraint in the quantum Hilbert space framework and conversely spinor algebra implies Lorentz invariance. Therefore the Lorentz group can be considered as a bridging algebraic structure between Quantum Information and Relativity Theory. Also the logic content of the associated linear algebra structures [3] could provide a new perspective to further explore this matter.

[1] Bakal, S ; Kim. Y. S. ; Noz, M. E., “Physics of the Lorentz Group”, IOP Publishing, Bristol, UK. 2015.

[2] Havel, T.F. and Doran, C.J.L., “Geometric algebra in quantum information processing”, in Quantum Computation and Information, Ed. Lomonaco, S.J. and Brandt, H.E., Contemp. Math., Volume 305, 2002.

[3] Toffano, Z. ; Dubois, F. “Adapting Logic to Physics : The Quantum-Like Eigenlogic Program”, Entropy, 22, 139, 2020.

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## Minkowski metric matrix entangling power and quantum concurrence

The Minkowski metric is represented by the diagonal matrix $\boldsymbol{G}$ acting on space-time quadrivectors:

$$\boldsymbol{G} = \mathrm{diag}(+1,-1,-1,-1)$$

In the language of Quantum Information $\boldsymbol{G}$ can be viewed as a *quantum entangling gate*

it acts in a similar way as the *Control-Z* gate $\quad \boldsymbol{Cz} = \mathrm{diag}(+1,+1,+1,-1)$.

Considering a uniform positive normalized input 2–qubit state:

$$|++\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) = \frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle) = \frac{1}{2}(1,1,1,1)^T$$

$$\boldsymbol{G}|++\rangle = \boldsymbol{G}\frac{1}{2}(1,1,1,1)^T = \frac{1}{2}(1,-1,-1,-1)^T = \frac{1}{2}(|00\rangle-|01\rangle-|10\rangle-|11\rangle) = \frac{1}{\sqrt{2}}(|0-\rangle-|1+\rangle)$$

The resulting state is the *singlet Bell state* for the 2-qubit basis: $\{|0+\rangle,|0-\rangle,|1+\rangle,|1-\rangle\}$

The *quantum concurrence* $C$ measures the degree of entanglement,

for a general 2-qubit state: $|\varphi\rangle = (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)$

The expression of concurrence is: $\quad C(|\varphi\rangle) = 2|ad-bc|$

For the state $\boldsymbol{G}|++\rangle$ concurrence is: $C\left(\frac{1}{2}(1,-1,-1,-1)^T\right) = 1$, indicating that it is *maximally entangled*.

## Lorentz Group and Minkowski Metric as Linear Operators of Quantum Information

- A Lorentz group matrix corresponds to a linear operator acting on a 2-qubit state and the Minkowski metric matrix of Relativity Theory can be considered an entangling 2-qubit gate.
- A non-relativistic 2-spinor can be identified with a 1-qubit state [2]. Rank-2 spinors, as Dirac 4-spinors, can be identified with a 2-qubit state, and so on…
- The Minkowski metric, according to the positive energy theorem in General Relativity, represents the ground state of the metric field. Its maximally entangled character analogous to the singlet Bell state suggests that entanglement and hyperbolic space are intimately interconnected.
- Quantum Information associated with the logic linear algebraic structures, as proposed in *Eigenlogic* [3], could provide a new tool to revisit the geometric structures (rotational and hyperbolic) of the Lorentz group linking Quantum Mechanics and Relativity Theory.

## References

[1] Bakal, S; Kim. Y. S.; Noz, M. E., “Physics of the Lorentz Group”, IOP Publishing, Bristol, UK. 2015

[2] Havel, T.F. and Doran, C.J.L., “ Geometric algebra in quantum information processing ”, in Quantum Computation and Information, Ed. Lomonaco, S.J. and Brandt, H.E., Contemp. Math., Volume 305, 2002

[3] Toffano, Z.; Dubois, F. “Adapting Logic to Physics: The Quantum-Like Eigenlogic Program”. Entropy, 22, 139, 2020

## Lorentz group generators and rotations

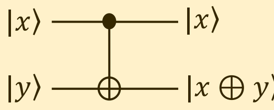
Rotation generators of the Lorentz group are represented by $4\times4$ matrices:

$$\boldsymbol{J}_1 = \begin{pmatrix}0&0&0&0\\0&0&0&i\\0&0&0&0\\0&-i&0&0\end{pmatrix},\ \boldsymbol{J}_2 = \begin{pmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{pmatrix},\ \boldsymbol{J}_3 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}$$

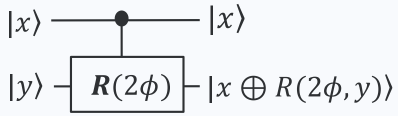
A rotation in the Lorentz group can be expressed as [1]: $\boldsymbol{Z}_3(\phi) = \exp(-i\phi\boldsymbol{J}_3) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\phi&-\sin\phi\\0&0&\sin\phi&\cos\phi\end{pmatrix}$

## Lorentz group rotation as a controlled 2-qubit gate

Control-*NOT*




Control-$\boldsymbol{R}(2\phi)$ Lorentz rotation $\boldsymbol{Z}_3(\phi)$




One can decompose the generators $\boldsymbol{J}_i$ using the Kronecker product: $\boldsymbol{J}_3 = \begin{pmatrix}0&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \boldsymbol{\Pi}_1\otimes\sigma_y$

$\boldsymbol{\Pi}_1$ is the 1-qubit logical-1 projector $|1\rangle\langle 1|$ and $\sigma_y$ is a Pauli matrix.

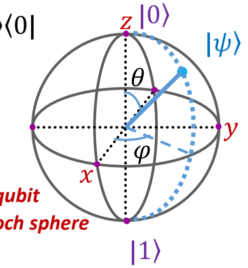
So one has: $\boldsymbol{Z}_3(\phi) = \exp\left(-i\phi(\boldsymbol{\Pi}_1\otimes\sigma_y)\right) = \boldsymbol{\Pi}_0\otimes\sigma_0 + \boldsymbol{\Pi}_1\otimes\exp(-i\phi\sigma_y) = \boldsymbol{\Pi}_0\otimes\sigma_0 + \boldsymbol{\Pi}_1\otimes\boldsymbol{R}_{\boldsymbol{y}}(2\phi)$

where $\exp(-i\phi\sigma_y) = \begin{pmatrix}\cos\phi&-\sin\phi\\\sin\phi&\cos\phi\end{pmatrix} = \boldsymbol{R}_{\boldsymbol{y}}(2\phi)$ is a 1-qubit rotation operator of angle $2\phi$ around $Oy$,

$\sigma_0 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ is the 1-qubit identity operator and $\boldsymbol{\Pi}_0$ is the 1-qubit logical-0 projector $|0\rangle\langle 0|$

$\boldsymbol{Z}_3(\phi)$ has a structure similar to the *Control-NOT* logical quantum gate $\boldsymbol{C}_{NOT}$ [3]

$$\boldsymbol{C}_{NOT} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} = \boldsymbol{\Pi}_0\otimes\sigma_0 + \boldsymbol{\Pi}_1\otimes\sigma_x = \boldsymbol{\Pi}_0\otimes\sigma_0 + \boldsymbol{\Pi}_1\otimes i\boldsymbol{R}_{\boldsymbol{x}}(\pi)$$




*1-qubit Bloch sphere*

where $\sigma_x = \begin{pmatrix}1&0\\0&1\end{pmatrix} = \boldsymbol{X}$ is the 1-qubit logical-*NOT* or *Pauli-X* gate, in Hilbert space it is a reflection or also a rotation of angle $\pi$ around the $Ox$ axis on the *Bloch sphere* multiplied by the imaginary number $i$.