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Games with distributionally robust joint chance constraints

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Abstract This paper studies an n-player non-cooperative game where each player has expected-value payoff function and chance-constrained strategy set. We consider the case where the row vectors defining the constraints are independent random vectors whose probability distributions are not completely known and belong to a certain distributional uncertainty set. The chance-constrained strategy sets are defined using a distributionally robust framework. We consider one density based uncertainty set and four two-moments based uncertainty sets. One of the considered uncertainty sets is based on a nonnegative support. Under the standard assumptions on the players' payoff functions, we show that there exists a Nash equilibrium of a distributionally robust chance-constrained game for each uncertainty set. As an application, we study Cournot competition in electricity market and perform the numerical experiments for the case of two electricity firms.

Keywords Chance-constrained game · Nash equilibrium · Distributionally robust optimization · Nonnegative support · Electricity market.

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1 Introduction

The theory of games was started with the minimax theorem by John von Neumann [24] which establishes the existence of a saddle point equilibrium in a zero-sum game. Later John Nash [23] showed that there exists a mixed strategy equilibrium, which is called a Nash equilibrium, for a finite strategic non-cooperative game with finite number of players. Since then, the non-cooperative strategic games have been extensively studied in the literature. The existence of a Nash equilibrium was shown under certain conditions on the strategy sets and payoff functions [2, 11, 13].

The above mentioned papers consider the games where players' strategy sets and payoff functions are deterministic in nature. In some practical game theoretic situations, the uncertainties are present due to various external factors. There are various ways to address the uncertainties present in the model. The robust optimization framework is used to handle distribution free uncertainties in the model [1, 34]. For the uncertainties involving random variables, the expected payoff criterion is used in case of risk neutral players [15, 16, 19, 26, 35, 36] and the risk measures CVaR and variance are used in the risk averse case [10, 18, 26]. For finite strategic games with random payoffs, Singh et al. [29, 30, 31] introduced a chance constraint programming based payoff criterion. It captures a situation where players are guaranteed to get the payoffs with a certain confidence level. There exists a mixed strategy Nash equilibrium of a chance-constrained game if the payoff vector of each player follows a multivariate elliptically symmetric distribution [29]. Such a Nash equilibrium can be computed by solving an equivalent mathematical program [31]. The characterization of the set of Nash equilibria of a chance-constrained game using the solution set of a variational inequality is given in [32]. The games where the probability distributions of players' payoffs are partially known is studied using distributionally robust approach [30]. The authors showed the existence of a mixed strategy Nash equilibrium for two different two-moments based uncertainty sets. For each uncertainty set, they proposed an equivalent mathematical program whose global maximizer gives a Nash equilibrium. There is a limited literature on zero-sum chance-constrained games, see for instance [3, 5, 6, 8].

Recently, the games with deterministic payoff functions and chance-constrained strategy set of each player have been introduced [25, 33]. In [33], the authors considered a two player zero-sum matrix game with individual chance constraints. For the case of elliptical distributions, they showed that a saddle point equilibrium of the game can be computed by solving a primal-dual pair of second order cone programs. In [25], the authors considered an n-player general sum game with joint chance constraints. The non-convexity of joint chance constraint is considered as a major difficulty for the existence of a Nash equilibrium. Peng et al. [25] proposed a new convex reformulation of the joint chance constraint when the row vectors defining random linear constraints are independent and follow multivariate normal distributions. Then, they showed that there exists a Nash equilibrium of a chance-constrained game (CCG). A brief summary of various chance-constrained game models studied by Singh et al. is given in Table 1.

In this paper, we consider an n-player non-cooperative game with expected value payoff functions and chance-constrained strategy sets defined in a distributionally robust framework. The random constraint vectors are independent and their probability distributions are not completely known. The only available information of a probability distribution is that it belongs to some distributional uncertainty set. We consider various types of distributional uncertainty sets which are constructed using partially available information of the underlying probability distribution. The first uncertainty set is based on the density of the random parameters where the normal distribution is the reference distribution. The distance between the true density function and the normal density function is defined using ϕ-divergence [17]. The other uncertainty sets are moments based, they are constructed with the information of mean vectors, covariance matrices and a support of a probability distribution [4, 7, 21]. One of the moments based uncertainty sets is defined on a nonnegative support. For ϕ-divergence uncertainty set, we show that a distributionally robust chance-constrained game (DRCCG) problem is equivalent to a CCG problem. Therefore, the existence of a Nash equilibrium in this case directly follows from the case of known probability distribution [25]. For each type of moments based uncertainty set, we propose a new convex reformulation of a joint chance constraint using a logarithmic transformation. Using convex reformulation of the players' strategy sets, we show that there exists a Nash equilibrium of a DRCCG under mild conditions on payoff functions. As an application, we consider a Cournot competition on electricity market comprising of a set of generation nodes and distribution nodes. The firms produce electricity at generation nodes and transmit to distribution nodes for the consumer. The transmission over long distances creates power losses and the firms want to keep their power losses below a certain threshold. The power losses are defined using random variables whose distributions are partially known. We model the transmission constraints as distributionally robust joint chance constraints. We performed numerical experiments by considering two electricity firms where a Nash equilibrium of the game is computed using a best response algorithm.

The games considered in this paper are significantly different from the ones considered in [30]. The players' payoff functions defined using distributionally robust individual chance constraints in [30] comes within the class of payoff functions considered in this paper. Unlike in [30], we consider the strategy set for each player defined by a distributionally robust joint chance constraint, and introduce two new moments based uncertainty sets, where one uncertainty set has a nonnegative support, and a density based uncertainty set.

The rest of the paper is organized as follows. Section 2 contains the definition of a DRCCG. Section 3 shows the existence of a Nash equilibrium of a DRCCG for different types of distributional uncertainty sets. Section 4 presents an application from an electricity market. We conclude the paper in Section 5.

Table 1: Existing chance-constrained game models

No.	Model	Payoffs	Strategy sets	Main Result
1	n-player finite strategic games considered in [29]	payoff functions are defined using a chance constraint	Mixed strategy setup	A Mixed strategy Nash equilibrium exists for elliptical distributions
2	2-player bi-matrix game considered in [31]	payoff functions are defined using a chance constraint	Mixed strategy setup	Equivalent mathematical program to compute Nash equilibrium for elliptical distributions
3	n-player finite strategic games considered in [30]	payoff functions are defined using a distributionally robust chance constraint	Mixed strategy setup	A Mixed strategy Nash equilibrium exists and it can be computed using equivalent mathematical program for two different two-moments based uncertainty sets
4	n-player continuous strategy games considered in [32]	payoff functions are defined using a chance constraint	A general convex and compact continuous strategy sets	For elliptically distributed payoffs, Nash equilibria and generalized Nash equilibria exists and are characterized using the solution sets of variational inequality
5	2- player zero-sum matrix game considered in [33]	payoff matrix is deterministic	strategy sets are defined using individual chance constraints	Saddle point equilibria are characterized using a primal-dual pair of second order cone programs when random constraint vectors follow elliptical distribution
6	n-player game considered in [25]	payoff functions are deterministic and satisfies standard continuity and concavity assumptions	Strategy sets are defined using a joint chance constraint	A Nash equilibrium exists when random constraint vectors are independent and follow multivariate normal distributions

2 The model

We consider an n-player non-cooperative game defined by the following objects:

- $I = \{1, 2, \cdots, n\}$ is the set of players.
- $X^i \subset \mathbb{R}^{m_i}_{++}$ denotes the set of all strategies of player i which is a convex and compact set; $\mathbb{R}^{m_i}_{++}$ ($\mathbb{R}^{m_i}_{+}$) denotes the positive (nonnegative) orthant of $\mathbb{R}^{m_i}$. The product set $X = \prod_{i \in I} X^i$ is the set of vectors of strategies of all the players and $X^{-i} = \prod_{j=1;j\neq i}^{n} X^j$ is the set of vectors of strategies of all the players but player i. A vector (y^i, x^{-i}) represents a strategy profile

where y^i is the strategy of player i and the strategy of player j, $j \neq i$, is x^j.

– Let $\xi : \Omega \to \mathbb{R}^d$ be a random vector defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $\omega \in \Omega$, let $v_i(x, \xi(\omega))$ represents a real valued payoff function of player i which is defined on $\prod_{i \in I} \mathbb{R}^{m_i}_{++} \times \mathbb{R}^d$. We consider the case where the probability distribution P_i of ξ is only partially known. The only information we have of P_i is that it belongs to a certain uncertainty set $\mathcal{P}_i$. We consider the worst-case scenario where the payoff function of player i, $i \in I$, is defined by
$$u_i(x) = \inf_{P_i \in \mathcal{P}_i} \mathbb{E}_{P_i}[v_i(x, \xi)],$$
where $\mathbb{E}_{P_i}$ denotes the expectation operator associated with probability distribution P_i.

We consider the case where the strategies of player i are further restricted by the following stochastic linear constraints
$$A^i x^i \leq b^i, \tag{1}$$
where $A^i = [A^i_1, A^i_2, \cdots, A^i_{K_i}]^T$ is a $K_i \times m_i$ random matrix defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $b^i \in \mathbb{R}^{K_i}$; T denotes the transposition. For each $k = 1, 2, \cdots, K_i$, A^i_k is the k^{th} row of A^i. We consider the case where the constraints of player i given by (1) are jointly satisfied with at least a given probability level α_i. Let F^i denote the probability distribution of A^i. In many practical situations, the probability distribution F^i is not completely known. Therefore, we consider the worst case scenario where constraints (1) are jointly satisfied with at least α_i probability for all possible distributions within a certain distributional uncertainty set $\mathcal{D}_i$. Then, the constraint (1) can be defined as a distributionally robust joint chance constraint given by
$$\inf_{F^i \in \mathcal{D}_i} \mathbb{P}\{A^i x^i \leq b^i\} \geq \alpha_i. \tag{2}$$
When $\mathcal{D}_i$, $i \in I$, contains only multivariate normal distribution, DRCCG reduces to a CCG considered in [25]. For an $\alpha_i \in [0,1]$, the feasible strategy set of player i is defined by
$$S^i_{\alpha_i} = \left\{ x^i \in X^i \mid \inf_{F^i \in \mathcal{D}_i} \mathbb{P}\{A^i x^i \leq b^i\} \geq \alpha_i \right\}, \ i \in I.$$
Using standard notations, we denote $S_\alpha = \prod_{i \in I} S^i_{\alpha_i}$ and $S^{-i}_{\alpha_{-i}} = \prod_{j \in I; j \neq i} S^j_{\alpha_j}$. We assume that the set $S^i_{\alpha_i}$ is non-empty, and the uncertainty sets $\mathcal{D}_i$, $i \in I$, and the probability level vector $\alpha = (\alpha_i)_{i \in I}$ are known to all the players. Then, the above DRCCG is a non-cooperative game with complete information. For a given α, a strategy profile x^* is said to be a Nash equilibrium of a DRCCG if and only if for each $i \in I$,
$$u_i(x^{i*}, x^{-i*}) \geq u_i(x^i, x^{-i*}), \ \forall \ x^i \in S^i_{\alpha_i}.$$
For the rest of the paper we have the following assumption on the players' payoff functions [22].

Assumption 1 *For each player i, $i \in I$, the following conditions hold.*

1. $v_i(x^i, x^{-i}, \xi)$ *is a concave function of* x^i *for every* $(x^{-i}, \xi) \in X^{-i} \times \mathbb{R}^d$.
2. $v_i(\cdot)$ *is a continuous function.*
3. $\mathbb{E}_{P_i}[v_i(x, \xi)]$ *is finite valued for any* $x \in X$ *and* $P_i \in \mathcal{P}_i$.
4. $\mathcal{P}_i$ *is weakly compact.*

3 Existence of Nash equilibrium for distributionally robust chance-constrained games

In this section, we consider the case where the rows of A^i are independent and a probability distribution of A^i is not completely known. We only know that it belongs to some uncertainty set which is constructed from the partially available information about the distribution. We consider five different types of uncertainty sets and for each case we show that there exists a Nash equilibrium of a DRCCG.

3.1 Density based uncertainty set

The uncertainty sets based on density function are often considered in the literature [17]. Such uncertainty sets are constructed by considering a reference distribution from estimated available data. The decision makers believe that their estimated distribution may not be correct but it is not very far from the true distribution. We assume a normal distribution as a reference distribution. We denote the normal density function of row vector A_k^i by f_{k0}^i. Then, the estimated density function f_0^i of A^i is also normal, and it is given by the product of the density functions $(f_{k0}^i)_{k=1}^{K_i}$. The distance between the estimated density f_0^i and the true density f^i is modeled by ϕ-divergence, which is defined as

$$D_i^{\phi}\left(f^i || f_0^i\right) = \int_{\mathbb{R}^{K_i m_i}} \phi\left(\frac{f^i(y)}{f_0^i(y)}\right) f_0^i(y) dy, \quad \forall\, i \in I. \tag{3}$$

The uncertainty set of player i, $i \in I$, is defined as

$$\mathcal{D}_i^{\phi} = \left\{ F^i \;\middle|\; D_i^{\phi}\left(f^i || f_0^i\right) \leq \varepsilon_i \right\},$$

where F^i is a probability distribution corresponding to the density function f^i and ε_i is divergence tolerance which represents the risk-aversion level of player i. Moreover, $\phi(s)$, called the ϕ-divergence function, is a convex function on $s \geq 0$. It can be extended to $\mathbb{R}$ by setting $\phi(s) = +\infty$ for $s < 0$. The ϕ-divergence function takes value zero if $f^i(y) > 0$ and $f_0^i(y) > 0$ have the same value, i.e., $\phi(1) = 0$. When the value of density function $f_0^i(y)$ is zero at some points, the terms $f_0^i(y)\phi\left(\frac{f^i(y)}{f_0^i(y)}\right)$ used in (3) are defined as $0\phi\left(\frac{s}{0}\right) := s \lim_{p \to +\infty} \phi(p)/p$ for $s > 0$, and $0\phi\left(\frac{0}{0}\right) := 0$.

Let ϕ^* be a conjugate function of ϕ defined as $\phi^*(t) = \sup_{s\in\mathbb{R}}\{ty - \phi(s)\}$. Define, $\underline{m}(\phi^*) = \sup\{m \in R : \phi^* \text{ is a finite constant on } (-\infty, m]\}$ and $\overline{m}(\phi^*) = \inf\{m \in R : \phi^*(m) = +\infty\}$. Let F_0^i be a probability distribution corresponding to density function f_0^i. Then, from Theorem 1 of [17] we have

$$\inf_{F^i\sim\mathcal{D}_i^\phi} \mathbb{P}\left\{A^i x^i \le b^i\right\} \ge \alpha_i = \mathbb{P}_{A^i\sim F_0^i}\left\{A^i x^i \le b^i\right\} \ge \hat{\alpha}_i, \tag{4}$$

where $\hat{\alpha}_i = \min\{\alpha_i', 1\}$ such that

$$\alpha_i' = \inf_{z>0,\, \underline{m}(\phi^*) \le z_0 + z \le \overline{m}(\phi^*)} \left\{ \frac{\phi^*(z_0+z) - z_0 - (1-\alpha_i)z + \varepsilon_i}{\phi^*(z_0+z) - \phi^*(z_0)} \right\}. \tag{5}$$

In (4), $A^i \sim F_0^i$ denotes that F_0^i is a probability distribution of A^i and the probability on the right hand side is defined according to F_0^i.

The values of $\underline{m}(\phi^*)$ and $\overline{m}(\phi^*)$ for certain ϕ-divergences are given in Table 1 of [17]. We summarize the values of α_i' for some famous ϕ-divergences obtained from (5) in Table 2. For detailed proofs for these α_i' we refer the readers to the Appendices of Jiang and Guan [17]. By using (4), the strategy

Table 2: The values α_i' for various ϕ-divergences

S.No.	$\phi(y)$, $y \ge 0$	α_i	α_i'
1	$(y-1)^2$	$(0.5, 1]$	$\alpha_i + \frac{\sqrt{\varepsilon_i^2 + 4\varepsilon_i\alpha_i(1-\alpha_i)} - (2\alpha_i - 1)\varepsilon_i}{2\varepsilon_i + 2}$
2	$\lvert y-1\rvert$	$[0, 1]$	$\alpha_i + \frac{\varepsilon_i}{2}$
3	$y\log y - y + 1$	$[0, 1]$	$\inf_{y\in(0,1)} \left\{\frac{e^{-\varepsilon_i} y^{\alpha_i} - 1}{y-1}\right\}$

set $S_{\alpha_i}^i$ of player i, $i \in I$, can be written as

$$S_{\alpha_i}^i = \left\{x^i \in X^i \mid \mathbb{P}_{A^i\sim F_0^i}\left\{A^i x^i \le b^i\right\} \ge \hat{\alpha}_i\right\}, \tag{6}$$

where $\hat{\alpha}_i$ can be obtained using (5). Under Assumption 1, the payoff function $u_i(x^i, x^{-i})$, $i \in I$, is a concave function of x^i for every x^{-i}. The strategy set $S_{\alpha_i}^i$, $i \in I$, defined by (6) has a convex and compact reformulation [25]. We further assume that $u_i(x^i, x^{-i})$, $i \in I$, is a non-increasing function of x^i for every x^{-i}. Then, it follows from Theorem 3.14 of [25] that there exists a Nash equilibrium of a DRCCG.

3.2 Moments based uncertainty set

We often encounter practical situations where we only have some information about the first two moments of underlying probability distribution. We consider four different two-moments based uncertainty sets and for each uncertainty set, we propose a new convex reformulation of the distributionally robust joint chance constraint (2). Under the convex reformulation, we show that there exists a Nash equilibrium of a DRCCG.

3.2.1 Uncertainty set with known first two order moments

We consider the uncertainty set of player i, $i \in I$, which accounts for the information about a mean vector μ_k^i and a covariance matrix Σ_k^i of $(A_k^i)^T$ for all $k = 1, 2, \cdots, K_i$. For each $i \in I$, define

$$\mathcal{D}_i^k(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \,\middle|\, \begin{array}{l} \mathbb{E}_{F_k^i}[(A_k^i)^T] = \mu_k^i, \\ \mathbb{E}_{F_k^i}[((A_k^i)^T - \mu_k^i)((A_k^i)^T - \mu_k^i)^T] = \Sigma_k^i \end{array} \right\}, \; k = 1, \cdots, K_i, \tag{7}$$

where F_k^i is a probability distribution of A_k^i and $\mathbb{E}_{F_k^i}$ is the expectation operator associated with F_k^i. Under independent assumption on the row vectors of matrix A^i [9], the distributionally robust joint chance constraint (2) is satisfied if and only if there exists a vector $z^i \in \mathbb{R}^{K_i}$ such that $\sum_{k=1}^{K_i} z_k^i = 1$, $z_k^i \geq 0$ for all $k = 1, 2, \cdots, K_i$ and

$$\inf_{F_k^i \in \mathcal{D}_i^k(\mu_k^i, \Sigma_k^i)} \mathbb{P}\{A_k^i x^i \leq b_k^i\} \geq \alpha_i^{z_k^i}, \quad k = 1, 2, \cdots K_i. \tag{8}$$

Then, using the deterministic reformulation of distributionally robust individual chance constraints (8) from [4, 12], we have the following deterministic reformulation for the distributionally robust joint chance constraint (2)

$$Q_{\alpha_i}^i = \begin{cases} (i) \; (\mu_k^i)^T x^i + \sqrt{\dfrac{\alpha_i^{z_k^i}}{1 - \alpha_i^{z_k^i}}} \left\| \left(\Sigma_k^i\right)^{1/2} x^i \right\| \leq b_k^i, \; \forall \; k = 1, 2, \cdots, K_i, \\ (ii) \; \displaystyle\sum_{k=1}^{K_i} z_k^i = 1, \\ (iii) \; z_k^i \geq 0, \; \forall \; k = 1, 2, \cdots, K_i, \end{cases} \tag{9}$$

where $||\cdot||$ is the Euclidean norm. It is clear that $Q_{\alpha_i}^i$ is not a convex set due to constraint (i) of (9). We reformulate the constraint (i) of (9) by using a change of variables technique under logarithmic transformation. We transform the vector $x^i \in X^i$ into a vector $y^i \in \mathbb{R}^{m_i}$, where $y_j^i = \ln x_j^i$, $j = 1, 2, \ldots, m_i$ [20]. Under logarithmic transformation, we have the following reformulation of (9)

$$\widetilde{Q}_{\alpha_i}^i = \begin{cases} (i) \; (\mu_k^i)^T e^{y^i} + \left\| \left(\Sigma_k^i\right)^{1/2} e^{\frac{1}{2}\left(z_k^i \log \alpha_i - \log\left(1 - \alpha_i^{z_k^i}\right)\right) \cdot \mathbb{1}_{m_i} + y^i} \right\| \leq b_k^i, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \; k = 1, 2, \cdots, K_i, \\ (ii) \; \displaystyle\sum_{k=1}^{K_i} z_k^i = 1, \\ (iii) \; z_k^i \geq 0, \; \forall \; k = 1, 2, \ldots, K_i. \end{cases} \tag{10}$$

Let Y^i be an image of X^i under logarithmic transformation. Since, the logarithmic function is continuous and X^i is a compact set, Y^i is also a compact set. The convexity need not be preserved under logarithmic transformation. From now onward, we consider the set X^i for which the set Y^i remains convex. Such sets indeed exists, see for instance [25]. The reformulation of feasible strategy set $S^i_{\alpha_i}$ of player i, $i \in I$, is given by

$$\widetilde{S}^i_{\alpha_i} = \left\{ (y^i, z^i) \in Y^i \times \mathbb{R}^{K_i} \mid (y^i, z^i) \in \widetilde{Q}^i_{\alpha_i} \right\}, \tag{11}$$

and it is a compact set.

Assumption 2 *For each $i \in I$ and $k = 1, 2, \ldots, K_i$, all the components of Σ^i_k and μ^i_k are nonnegative.*

Under Assumption 2, we show that the set $\widetilde{S}^i_{\alpha_i}$ is convex. It suffices to show that the constraint (i) of (10) is convex.

Lemma 1 *For each $i \in I$, the set $\widetilde{S}^i_{\alpha_i}$ is a convex set, under Assumption 2, for all $\alpha_i \in (0, 1)$.*

Proof Fix $\alpha_i \in (0, 1)$. The function $\log(1 - p)$ is non-increasing and concave in p, and $\alpha_i^{z^i_k}$ is a convex function of z^i_k. Therefore, the composition function $\log\left(1 - \alpha_i^{z^i_k}\right)$ is a concave function of z^i_k. Then, $\left(e^{\frac{1}{2}\left(z^i_k \log \alpha_i - \log\left(1 - \alpha_i^{z^i_k}\right)\right) \cdot \mathbb{1}_{m_i} + y^i}\right)$ is an $m_i \times 1$ vector whose components are nonnegative convex functions. Therefore, it follows from Proposition 2.1 of [14] that $\left\| \left(\Sigma^i_k\right)^{1/2} e^{\frac{1}{2}\left(z^i_k \log \alpha_i - \log\left(1 - \alpha_i^{z^i_k}\right)\right) \cdot \mathbb{1}_{m_i} + y^i} \right\|$ is a convex function. The term $(\mu^i_k)^T e^{y^i}$ is a convex function because μ^i_k is a nonnegative vector. Hence, the constraints

$$(\mu^i_k)^T e^{y^i} + \left\| \left(\Sigma^i_k\right)^{1/2} e^{\frac{1}{2}\left(z^i_k \log \alpha_i - \log\left(1 - \alpha_i^{z^i_k}\right)\right) \cdot \mathbb{1}_{m_i} + y^i} \right\| \leq b^i_k, \; \forall\, k = 1, 2, \cdots, K_i,$$

are convex. It is easy to see that the other constraints of $\widetilde{S}^i(\alpha_i)$ are convex. Therefore, $\widetilde{S}^i_{\alpha_i}$, $i \in I$ is a convex set. □

3.2.2 Uncertainty set with unknown second order moment

In this section, we consider the uncertainty set of player i, $i \in I$, which accounts for the information about a mean vector μ^i_k and an upper bound $\Sigma^i_k \succ 0$ on covariance matrix of $(A^i_k)^T$ for all $k = 1, 2, \cdots, K_i$. For each $i \in I$, define

$$\mathcal{D}^k_i(\mu^i_k, \Sigma^i_k) = \left\{ F^i_k \;\middle|\; \begin{array}{l} \mathbb{E}_{F^i_k}[(A^i_k)^T] = \mu^i_k, \\ \mathbb{E}_{F^i_k}[((A^i_k)^T - \mu^i_k)((A^i_k)^T - \mu^i_k)^T] \preceq \Sigma^i_k \end{array} \right\}, k = 1, 2, \cdots, K_i. \tag{12}$$

Under independent assumption on the row vectors of A^i, it follows from [7] that the deterministic reformulation of distributionally robust joint chance constraint (2) for uncertainty set (7) is the same as for the uncertainty set (12) and it is given by (9). Therefore, under logarithmic transformation described in Section 3.2.1, a convex and compact reformulation of feasible strategy set $S^i_{\alpha_i}$, $i \in I$, is given by (11).

3.2.3 Uncertainty set with unknown moments

We consider the case where the mean vector of $(A^i_k)^T$ lies in an ellipsoid of size $\gamma^i_{k1} \geq 0$ centered at μ^i_k and the covariance matrix of $(A^i_k)^T$ lies in a positive semi-definite cone defined with a linear matrix inequality. For each $i \in I$ and $k = 1, 2 \cdots, K_i$, let $\Sigma^i_k \succ 0$, $\gamma^i_{k2} > 0$. We define the uncertainty set of each player $i \in I$ as follows

$$\mathcal{D}^k_i(\mu^i_k, \Sigma^i_k) = \left\{ F^i_k \middle| \begin{array}{l} \left(\mathbb{E}_{F^i_k}[(A^i_k)^T] - \mu^i_k\right)^\top \left(\Sigma^i_k\right)^{-1} \left(\mathbb{E}_{F^i_k}[(A^i_k)^T] - \mu^i_k\right) \leq \gamma^i_{k1}, \\ COV_{F^i_k}[(A^i_k)^T] \preceq \gamma^i_{k2} \Sigma^i_k \end{array} \right\} \tag{13}$$

for all $k = 1, \cdots, K_i$. $COV_{F^i_k}$ is a covariance operator under probability distribution F^i_k. As mentioned earlier in Section 3.2.1, under independent assumption the chance constraint (2) can be equivalently written as

$$\inf_{F^i_k \in \mathcal{D}^k_i(\mu^i_k, \Sigma^i_k)} \mathbb{P}\left\{A^i_k x^i \leq b^i_k\right\} \geq \alpha_i^{z^i_k}, \tag{14}$$
$$\sum_{k=1}^{K_i} z^i_k = 1, \ z^i_k \geq 0, \ k = 1, 2, \cdots, K_i.$$

Based on the structure of the uncertainty set $\mathcal{D}^k_i(\mu^i_k, \Sigma^i_k)$, the constraint (14) can be written as

$$\inf_{(\mu,\Sigma) \in \mathcal{U}^i_k} \inf_{F^i_k \in \mathcal{D}(\mu,\Sigma)} \mathbb{P}\left\{A^i_k x^i \leq b^i_k\right\} \geq \alpha_i^{z^i_k},$$

where

$$\mathcal{D}(\mu, \Sigma) = \left\{ F^i_k \middle| \mathbb{E}_{F^i_k}[(A^i_k)^T] = \mu, COV_{F^i_k}[A^i_k] = \Sigma \right\},$$

and

$$\mathcal{U}^i_k = \left\{ (\mu, \Sigma) \middle| \left(\mu - \mu^i_k\right)^\top \left(\Sigma^i_k\right)^{-1} \left(\mu - \mu^i_k\right) \leq \gamma^i_{k1}, \Sigma \preceq \gamma^i_{k2} \Sigma^i_k \right\}.$$

According to one-sided Chebyshev inequality [21, 27], we have

$$\inf_{F^i_k \in \mathcal{D}(\mu,\Sigma)} \mathbb{P}\left\{A^i_k x^i \leq b^i_k\right\} = \begin{cases} 1 - \frac{1}{1 + \frac{(\mu^T x^i - b^i_k)^2}{((x^i)^T \Sigma x^i)}}, & \text{if } \ \mu^T x^i \leq b^i_k, \\ 0, & \text{otherwise.} \end{cases}$$

For the case $\mu^T x^i > b_k^i$,

$$\inf_{F_k^i \in \mathcal{D}(\mu,\Sigma)} \mathbb{P}\left\{A_k^i x^i \leq b_k^i\right\} = 0,$$

and it leads constraint (14) to be infeasible. When $\mu^T x^i \leq b_k^i$, the constraint (14) is equivalent to

$$\inf_{(\mu,\Sigma)\in\mathcal{U}_k^i} 1 - \frac{1}{1+(\mu^T x^i - b_k^i)^2/((x^i)^T \Sigma x^i)} \geq \alpha_i^{z_k^i},$$

which can be reformulated as

$$h_k^i(x^i) \geq \sqrt{\frac{\alpha_i^{z_k^i}}{1-\alpha_i^{z_k^i}}}, \tag{15}$$

where

$$h_k^i(x^i) = \begin{cases} \min\limits_{\mu,\Sigma} \frac{b_k^i - \mu^T x^i}{\sqrt{(x^i)^T \Sigma x^i}} \\ \text{s.t. } (i)\ \left(\mu - \mu_k^i\right)^\top \left(\Sigma_k^i\right)^{-1} \left(\mu - \mu_k^i\right) \leq \gamma_{k1}^i, \\ (ii)\ \Sigma \preceq \gamma_{k2}^i \Sigma_k^i. \end{cases} \tag{16}$$

The problem (16) can be separated into two optimization problems. Therefore,

$$h_k^i(x^i) = \frac{b_k^i + v_1(x^i)}{\sqrt{v_2(x^i)}},$$

where

$$v_1(x^i) = \begin{cases} \min\limits_{\mu} -\mu^T x^i \\ \text{s.t. } \ \left(\mu - \mu_k^i\right)^\top \left(\Sigma_k^i\right)^{-1} \left(\mu - \mu_k^i\right) \leq \gamma_{k1}^i, \end{cases} \tag{17}$$

$$v_2(x^i) = \begin{cases} \max\limits_{\Sigma}\ (x^i)^T \Sigma x^i \\ \text{s.t. } \Sigma \preceq \gamma_{k2}^i \Sigma_k^i. \end{cases}$$

Let $\lambda \geq 0$ be a Lagrange multiplier associated with the constraint of optimization problem (17). By applying the KKT conditions, the optimal solution of (17) is given by $\mu = \mu_k^i + \frac{\sqrt{\gamma_{k1}^i}\Sigma_k^i x^i}{\sqrt{(x^i)^T \Sigma_k^i x^i}}$ and associated Lagrange multiplier is given by $\lambda = \sqrt{\frac{(x^i)^T \Sigma_k^i x^i}{4\gamma_{k1}^i}}$. Therefore, the corresponding optimal value $v_1(x^i) = -(\mu_k^i)^T x^i - \sqrt{\gamma_{k1}^i}\sqrt{(x^i)^T \Sigma_k^i x^i}$. Since, $u^T \Sigma u \leq u^T \gamma_{k2}^i \Sigma_k^i u$ for any $u \in \mathbb{R}^n$, then, $v_2(x^i) = \gamma_{k2}^i (x^i)^T \Sigma_k^i x^i$. Using this, constraint (15) can be reformulated as

$$(\mu_k^i)^T x^i + \left(\sqrt{\frac{\alpha_i^{z_k^i}}{1-\alpha_i^{z_k^i}}}\sqrt{\gamma_{k2}^i} + \sqrt{\gamma_{k1}^i}\right) \left\| \left(\Sigma_k^i\right)^{1/2} x^i \right\| \leq b_k^i. \tag{18}$$

Hence, we have the following deterministic reformulation for the distributionally robust joint chance constraint (2)

$$Q^i_{\alpha_i} = \begin{cases} (i)\ (\mu^i_k)^T x^i + \left(\sqrt{\dfrac{\alpha_i^{z^i_k}}{1-\alpha_i^{z^i_k}}} \sqrt{\gamma^i_{k2}} + \sqrt{\gamma^i_{k1}} \right) \left\| \left(\Sigma^i_k\right)^{1/2} x^i \right\| \leq b^i_k, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\ k = 1, 2, \cdots, K_i, \\ (ii)\ \sum_{k=1}^{K_i} z^i_k = 1, \\ (iii)\ z^i_k \geq 0,\ \forall\ k = 1, 2, \cdots, K_i. \end{cases} \tag{19}$$

For a convex reformulation of (19), we use logarithmic transformation described in Section 3.2.1. The new convex reformulation of (19) is given by

$$\widetilde{Q}^i_{\alpha_i} = \begin{cases} (i)\ (\mu^i_k)^T e^{y^i} + \sqrt{\gamma^i_{k2}} \left\| \left(\Sigma^i_k\right)^{1/2} e^{\frac{1}{2}\left(z^i_k \log \alpha_i - \log\left(1-\alpha_i^{z^i_k}\right)\right)\cdot \mathbb{1}_{m_i} + y^i} \right\| \\ \qquad\qquad + \sqrt{\gamma^i_{k1}} \left\| \left(\Sigma^i_k\right)^{1/2} e^{y^i} \right\| \leq b^i_k,\ \forall\ k = 1, 2, \cdots, K_i, \\ (ii)\ \sum_{k=1}^{K_i} z^i_k = 1, \\ (iii)\ z^i_k \geq 0,\ \forall\ k = 1, 2, \cdots, K_i. \end{cases} \tag{20}$$

Hence, the reformulation of feasible strategy set $S^i_{\alpha_i}$ of player i, $i \in I$, is given by

$$\widetilde{S}^i_{\alpha_i} = \left\{ (y^i, z^i) \in Y^i \times \mathbb{R}^{K_i} \mid (y^i, z^i) \in \widetilde{Q}^i_{\alpha_i} \right\}.$$

The reformulation (20) is similar to (10) except in constraint (i) where there is one extra term and a positive multiplier in the second term. The convexity of (20) follows from the similar arguments used in Lemma 1. It is also a compact set. Therefore, the reformulated feasible strategy set $\widetilde{S}^i_{\alpha_i}$ of player i for distributional uncertainty set defined by (13) is a convex and compact set.

3.2.4 Uncertainty set with known first order moment and nonnegative support

So far we have considered the full support for the random vector A^i_k in the uncertainty sets. However, in some cases the random vector A^i_k can be nonnegative. To the best of our knowledge, distributionally robust games with nonnegative support have not been considered so far in the literature. To maintain the feasibility of the chance constraint (2), we further assume that $b^i_k > 0$ for all $k = 1, 2, \cdots, K_i$. We define the uncertainty set for player i, $i \in I$, as follows

$$\mathcal{D}^k_i(\mu^i_k) = \left\{ F^i_k \mid \mathbb{E}_{F^i_k}[(A^i_k)^T] = \mu^i_k, \mathbb{P}_{F^i_k}[A^i_k \geq 0] = 1 \right\},\ k = 1, \ldots, K_i, \tag{21}$$

where $\mu_k^i \geq 0$ for all $k = 1, \cdots, K_i$. Under independent assumption the chance constraint (2) can be equivalently written as

$$\left.\begin{aligned} &\inf_{F_k^i \in \mathcal{D}_i^k(\mu_k^i)} \mathbb{P}\left\{A_k^i x^i \leq b_k^i\right\} \geq z_k^i,\ k = 1, 2, \cdots, K_i \\ &\prod_{k=1}^{K_i} z_k^i \geq \alpha_i,\ 0 \leq z_k^i \leq 1,\ k = 1, 2, \cdots, K_i. \end{aligned}\right\} \tag{22}$$

For each $k = 1, 2, \cdots, K_i$, consider the optimization problem $\inf_{F_k^i \in \mathcal{D}_i^k(\mu_k^i)} \mathbb{P}\left\{A_k^i x^i \leq b_k^i\right\}$ which can be reformulated as

$$\begin{aligned} \inf_{F_k^i} &\int_{A_k^i \geq 0} \mathbf{1}_{A_k^i x^i \leq b_k^i} \mathrm{d}F_k^i(A_k^i) \\ \text{s.t.} &\int_{A_k^i \geq 0} (A_k^i)^T \mathrm{d}F_k^i(A_k^i) = \mu_k^i \\ &\int_{A_k^i \geq 0} \mathrm{d}F_k^i(A_k^i) = 1 \end{aligned} \tag{23}$$

where $\mathbf{1_C}$ is the indicator function over a given set $\mathbf{C}$. The dual problem of (23) is given by

$$\begin{aligned} \sup_{\beta_k^i \in \mathbb{R}^{m_i}, \lambda_k^i \in \mathbb{R}} \quad & (\mu_k^i)^T \beta_k^i + \lambda_k^i \\ \text{s.t.} \quad & \mathbf{1}_{A_k^i x^i \leq b_k^i} - A_k^i \beta_k^i - \lambda_k^i \geq 0, \forall A_k^i \geq 0. \end{aligned} \tag{24}$$

The strong duality follows from [28] because Dirac measure $\delta_{\mu_k^i}$ lies in the relative interior of $\mathcal{D}_i^k(\mu_k^i)$. The constraint (24) can be reformulated as

$$A_k^i \beta_k^i + \lambda_k^i \leq 1, \forall A_k^i \geq 0, \tag{25}$$
$$A_k^i \beta_k^i + \lambda_k^i \leq 0, \forall A_k^i \geq 0, A_k^i x^i > b_k^i. \tag{26}$$

Constraint (25) is equivalent to $\beta_k^i \leq 0$ and $\lambda_k^i \leq 1$. Since, we look for the supremum in (24), we can replace $A_k^i x^i > b_k^i$ in (26) by $A_k^i x^i \geq b_k^i$. Then, using duality theory of linear programming, the constraint (26) can be equivalently written as

$$\begin{aligned} \lambda_k^i \leq b_k^i \tau_k^i, \tau_k^i \geq 0, \\ \beta_k^i + \tau_k^i x^i \leq 0. \end{aligned}$$

Since, there is no duality gap between (23) and (24), using (22) we have the following deterministic reformulation for the distributionally robust joint chance constraint (2)

$$Q^i_{\alpha_i} = \begin{cases} (i)\ (\mu^i_k)^T\beta^i_k + \lambda^i_k \geq z^i_k,\ k=1,2,\cdots,K_i, \\ (ii)\ \beta^i_k \leq 0, \lambda^i_k \leq 1,\ k=1,2,\cdots,K_i, \\ (iii)\ \lambda^i_k \leq b^i_k\tau^i_k, \tau^i_k \geq 0,\ k=1,2,\cdots,K_i, \\ (iv)\ \beta^i_k + \tau^i_k x^i \leq 0,\ k=1,2,\cdots,K_i, \\ (v)\ \prod_{k=1}^{K_i} z^i_k \geq \alpha_i,\ 0 \leq z^i_k \leq 1,\ k=1,2,\cdots,K_i \end{cases} \tag{27}$$

Since $\mu^i_k \geq 0$, $\beta^i_k \leq 0$ and $0 \leq z^i_k \leq 1$, constraint (i) in (27) implies $\lambda^i_k \geq 0$. In fact, it is easy to see that the components of vector $(\tau^i_k, \lambda^i_k, z^i_k, -\beta^i_k)$ are positive. Therefore, we use change of variables under logarithmic transformation as follows: $\tilde{\lambda}^i_k = \log(\lambda^i_k)$, $\tilde{\beta}^i_{kj} = \log(-\beta^i_{kj})$, $\tilde{\tau}^i_k = \log(\tau^i_k)$, $\zeta^i_k = \log(z^i_k)$, and $y^i_j = \ln x^i_j$, for all $j = 1,2,\ldots,m_i$, $k = 1,2,\ldots,K_i$. We have a new convex reformulation of (27) given by

$$\widetilde{Q}^i_{\alpha_i} = \begin{cases} (i)\ e^{\zeta^i_k - \tilde{\lambda}^i_k} + (\mu^i_k)^T e^{\tilde{\beta}^i_k - \tilde{\lambda}^i_k \mathbf{1}_{m_i}} \leq 1,\ k=1,2,\cdots,K_i, \\ (ii)\ \tilde{\lambda}^i_k \leq 0,\ k=1,2,\cdots,K_i, \\ (iii)\ \tilde{\lambda}^i_k \leq \tilde{\tau}^i_k + \log(b^i_k),\ k=1,2,\cdots,K_i, \\ (iv)\ \tilde{\tau}^i_k + y^i_j - \tilde{\beta}^i_{kj} \leq 0,\ k=1,2,\cdots,K_i,\ j=1,\cdots,m_i, \\ (v)\ \sum_{k=1}^{K_i} \zeta^i_k \geq \log(\alpha_i), \zeta^i_k \leq 0,\ k=1,2,\cdots,K_i \end{cases}$$

Hence, the reformulation of feasible strategy set $S^i_{\alpha_i}$ of player i, $i \in I$, is given by

$$\widetilde{S}^i_{\alpha_i} = \left\{ (y^i, \tilde{\lambda}^i, \tilde{\beta}^i, \tilde{\tau}^i, \zeta^i) \in Y^i \times \mathbb{R}^{K_i} \times \mathbb{R}^{K_i \cdot m_i} \times \mathbb{R}^{K_i} \times \mathbb{R}^{K_i} \mid (y^i, \tilde{\lambda}^i, \tilde{\beta}^i, \tilde{\tau}^i, \zeta^i) \in \widetilde{Q}^i_{\alpha_i} \right\}.$$

It is evident that $\widetilde{S}^i_{\alpha_i}$ is a convex set. It follows from (27) that $Q^i_{\alpha_i}$ is a compact set. Then, under logarithmic transformation $\widetilde{S}^i_{\alpha_i}$, $i \in I$, is a compact set.

We present a general theorem on the existence of a Nash equilibrium for a DRCCG which covers the case of each uncertainty set as a special case. The concavity of a player's payoff function in its strategies, for a fixed strategy profile of other players, plays a crucial role in showing the existence of a Nash equilibrium. Therefore, we need the following assumption on players' payoff functions which ensures the required concavity property of the payoff functions under logarithmic transformation.

Assumption 3 *For each player i, $i \in I$, the payoff function $u_i(\cdot, x^{-i})$ satisfies the non-increasing condition for every $x^{-i} \in X^{-i}$, i.e., for any two points x^i and $\bar{x}^i$ such that $x^i_k \leq \bar{x}^i_k$ for all $k = 1,2,\ldots,m_i$, we have $u_i(x^i, x^{-i}) \geq u_i(\bar{x}^i, x^{-i})$.*

Theorem 1 *Consider an n-player DRCCG, where*

1. *the payoff function of player i, $i \in I$, satisfies the Assumptions 1 and 3,*
2. *the reformulation of feasible strategy set $S^i_{\alpha_i}$, $i \in I$, under logarithmic transformation is a convex and compact set.*

Then, there exists a Nash equilibrium of a DRCCG for all $\alpha \in (0,1)^n$.

Proof Let $\alpha \in (0,1)^n$. Under Assumption 1, the payoff function $u_i(x^i, x^{-i})$, $i \in I$, is a concave function of x^i for every $x^{-i} \in X^{-i}$, and a continuous function of x. For each $i \in I$, define a composition function $V_i = u_i \circ d_i$, where $d_i : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_n} \to \mathbb{R}^{m_1}_{++} \times \mathbb{R}^{m_2}_{++} \times \cdots \times \mathbb{R}^{m_n}_{++}$, such that

$$d_i\left(y^1, y^2, \cdots, y^n\right) = \left(e^{y^1}, e^{y^2}, \cdots, e^{y^n}\right).$$

Under Assumptions 3, the composition function $V_i(\cdot, y^{-i})$, $i \in I$, is a concave function of y^i for every y^{-i} and $V_i(\cdot)$ is a continuous function of y. Let $\widetilde{S}^i_{\alpha_i}$, $i \in I$, be the reformulation of the feasible strategy set $S^i_{\alpha_i}$ under logarithmic transformation which is a convex and compact set. Then, there exists a Nash equilibrium of an n-player non-cooperative game defined by strategy sets $\left(\widetilde{S}^i_{\alpha_i}\right)_{i\in I}$ and payoff functions $\left(V_i(\cdot)\right)_{i\in I}$ [11, 13]. Therefore, there exists $y^* \in \prod_{i\in I} \widetilde{S}^i_{\alpha_i}$ such that

$$V_i(y^{i*}, y^{-i*}) \geq V_i(y^i, y^{-i*}), \ \forall\ y^i \in \widetilde{S}^i_{\alpha_i}$$

Under the hypothesis of the theorem, $\widetilde{S}^i_{\alpha_i}$ is a reformulation of $S^i_{\alpha_i}$, where $x^i = e^{y^i}$. Therefore, for $x^* = e^{y^*}$ we have

$$u_i(x^{i*}, x^{-i*}) \geq u_i(x^i, x^{-i*}), \ \forall\ x^i \in S^i_{\alpha_i}.$$

Hence, x^* is a Nash equilibrium of a DRCCG for all $\alpha \in (0,1)^n$. □

The reformulated feasible strategy sets $\widetilde{S}^i_{\alpha_i}$, $i \in I$, is a convex and compact set for each uncertainty set. Therefore, the corollary given below follows directly from Theorem 1.

Corollary 1 *Consider an n-player DRCCG, where*

1. *the payoff function of player i, $i \in I$, satisfies the Assumptions 1 and 3,*
2. *the row vectors of A^i, $i \in I$ are independent,*
3. *one of the following conditions hold true*
   - *for each $k = 1, 2, \cdots K_i$, the probability distribution of row vector A^i_k belongs to uncertainty set $\mathcal{D}^k_i(\mu^i_k, \Sigma^i_k)$ defined as in (7), where the mean vector μ^i_k and the covariance matrix Σ^i_k satisfy Assumption 2.*

     or
   - *for each $k = 1, 2, \cdots K_i$, the probability distribution of row vector A^i_k belongs to uncertainty set $\mathcal{D}^k_i(\mu^i_k, \Sigma^i_k)$ defined as in (12), where mean vector μ^i_k and the upper bound Σ^i_k on covariance matrix satisfy Assumption 2.*

     or

- *for each* $k = 1, 2, \cdots K_i$, *the probability distribution of row vector* A_k^i *belongs to uncertainty set* $\mathcal{D}_i^k(\mu_k^i, \Sigma_k^i)$ *defined as in* (13)*, where* μ_k^i *and* Σ_k^i *satisfy Assumption 2.*
  or
- *for each* $k = 1, 2, \cdots K_i$, *the probability distribution of row vector* A_k^i *belongs to uncertainty set* $\mathcal{D}_i^k(\mu_k^i)$ *defined as in* (21)*.*

Then, there exists a Nash equilibrium of a DRCCG for all $\alpha \in (0, 1)^n$*.*

4 Cournot competition in electricity market

We consider an electricity market where the firms compete over an electricity network comprises of a set of nodes. There are several generation nodes where the firms installed their generation facilities to produce electricity. There are also some distribution nodes from where the electricity is distributed to the consumers. The firms generate the electricity at their facilities and transmit to the distribution nodes. The transmission over long distances creates power losses which is better modeled using random variables. For simplicity, we omit the problem of consumers, which means any quantity of electricity can be consumed. The components of electricity market are described as follows:

I- set of firms called as players

N – the set of generation nodes

N_i – subset of generation nodes where firm i has installed its generation facilities

I_k – the set of firms who owns generation facilities at node k

M – the set of distribution nodes

Let $x_k^i = (x_{kj}^i)_{j \in M}$ be a vector of quantities being transmitted from the generation node k to the distribution nodes by firm i, e.g., x_{kj}^i is the quantity transmitted from the generation node k to the distribution node j by firm i. Denote a generation level vector of firm i by $x^i = (x_k^i)_{k \in N_i}$. We assume that the price at a generation node k is different for each distribution node j and it depends on the total amount of electricity being transmitted to the node j from the node k. The price for a distribution node j at a generation node k is given by,

$$P_{kj} = \beta_{kj} - \delta_{kj} \sum_{i \in I_k} x_{kj}^i, \; k \in N, \; j \in M, \tag{28}$$

where $\delta_{kj} \geq 0$ for all $k \in N$, $j \in M$. Each firm incurs cost in generation as well as in the transmission of electricity. Let $c_{kj}^i(x_{kj}^i)$ be the cost incurred by firm i for the generation and transmission of x_{kj}^i unit of electricity from node k to node j. We consider the linear cost function for each firm, i.e., $c_{kj}^i(x_{kj}^i) = c_{kj}^i x_{kj}^i$ for all $k \in N_i$, $j \in M$, $i \in I$. The payoff function of firm i is given by

$$u_i(x^i, x^{-i}) = \sum_{k\in N_i}\sum_{j\in M}\left(x^i_{kj}\left(\beta_{kj} - \delta_{kj}\sum_{l\in I_k} x^l_{kj}\right) - c^i_{kj}x^i_{kj}\right). \tag{29}$$

The case where parameters β_{kj}, δ_{kj} and c^i_{kj} are random variables, e.g., when the market has randomness in the price and cost functions. In such cases, we take the average values of these parameters. The strategy set of player i is defined as $X^i = \{x^i = (x^i_k)_{k\in N_i} \mid x^i_{kj} \in [\varepsilon^i_{kj}, \gamma^i_{kj}], k \in N_i, j \in M\}$, where $\varepsilon^i_{kj} > 0$ and γ^i_{kj} denote the minimum and maximum output of firm i from node k to node j, respectively. Let $a^i_k = (a^i_{kj})_{j\in M}$ be a vector of unit electricity loss for firm i at generation node k where a^i_{kj} represents the unit electricity loss during the transmission from the kth generation node to the jth distribution node. We assume that a^i_{kj} is a random variable for each $i \in I$, $j \in M$ and $k \in N_i$. Each firm wants to keep its loss under a certain threshold. Let b^i denotes the threshold vector for firm i. For a given confidence level α_i, the distributionally robust joint chance constraint of firm i, $i \in I$, is given by

$$\inf_{F^i\in\mathcal{D}_i} \mathbb{P}\left\{(a^i_k)^T x^i_k \le b^i_k, k \in N_i\right\} \ge \alpha_i. \tag{30}$$

We assume that the electricity loss for each firm at different node is independent, i.e., the random vectors $(a^i_k)_{k\in N_i}$ are independent. Then, under the logarithmic transformation, the payoff function of player i, $i = 1, 2$, is given by

$$V_i(y^i, y^{-i}) = \sum_{k\in N_i}\sum_{j\in M}\left(e^{y^i_{kj}}\left(\beta_{kj} - \delta_{kj}\sum_{l\in I_k} e^{y^l_{kj}}\right) - c^i_{kj}e^{y^i_{kj}}\right). \tag{31}$$

For a fixed y^{-i}, the function $V_i(y^i, y^{-i})$ is a concave function of y^i if its Hessian matrix is negative semidefinite. The off-diagonal entries of the Hessian matrix are zero and the diagonal entries are given by

$$\frac{\partial^2 V_i(y^i, y^{-i})}{(\partial y^i_{kj})^2} = [(\beta_{kj} - \delta_{kj}\sum_{l\in I_k} e^{y^l_{kj}} - c^i_{kj}) \times e^{y^i_{kj}}] - [3 \times e^{2y^i_{kj}} \times \delta_{kj}].$$

Then, the Hessian matrix is negative semidefinite if

$$3e^{y^i_{kj}} + \sum_{l\in I_k} e^{y^l_{kj}} \ge \frac{\beta_{kj} - c^i_{kj}}{\delta_{kj}},\ k \in N_i\ j \in M. \tag{32}$$

The condition 32 holds for the suitable choice of a minimum output, ε^i_{kj}, of the firms . The strategy set, under logarithmic transformation, is given by

$$Y^i = \left\{y^i = (y^i_{kj})_{k\in N_i, j\in M} \mid y^i_{kj} \in [\ln\varepsilon^i_{kj}, \ln\gamma^i_{kj}],\ k \in N_i, j \in M\right\}.$$

It is clear that Y^i is a convex and compact set. We consider the case of known mean μ^i_k and known co-variance matrix Σ^i_k of a^i_k. In this case, the reformulated feasible strategy set $\tilde{S}^i_{\alpha_i}$ is given by (11).

4.1 Case Study

For illustration purpose, we consider the case of two electricity firms whose generation nodes are the same. It is easy to see that the condition (32) holds if

$$\varepsilon_{kj}^1 \geq \frac{3\beta_{kj} - 4c_{kj}^1 + c_{kj}^2}{15\delta_{kj}},\ \varepsilon_{kj}^2 \geq \frac{3\beta_{kj} - 4c_{kj}^2 + c_{kj}^1}{15\delta_{kj}},\ k \in N, j \in M. \tag{33}$$

Therefore, it follows from Corollary 1 that there exists a Nash equilibrium of the game. For a fixed $(y^2, z^2) \in \tilde{S}_{\alpha_2}^2$, player 1 solves the following convex optimization problem

$$\begin{aligned} [P_1] \quad & \max_{y^1, z^1} V_1(y^1, y^2) \\ \text{s.t.} \quad & (y^1, z^1) \in \widetilde{S}_{\alpha_1}^1. \end{aligned}$$

The set of optimal solution of $[P_1]$, which is also called the best response set of player 1, is given by

$$BR_1(y^2) = \{(\bar{y}^1, \bar{z}^1) \mid V_1(\bar{y}^1, y^2) \geq V_1(y^1, y^2),\ \forall\ (y^1, z^1) \in \widetilde{S}_{\alpha_1}^1\}.$$

Similarly, for a fixed $(y^1, z^1) \in \tilde{S}_{\alpha_1}^1$, player 2 solves the following convex optimization problem

$$\begin{aligned} [P_2] \quad & \max_{y^2, z^2} V_2(y^1, y^2) \\ \text{s.t.} \quad & (y^2, z^2) \in \widetilde{S}_{\alpha_2}^2. \end{aligned}$$

The best response set of player 2, is given by

$$BR_2(y^1) = \{(\bar{y}^2, \bar{z}^2) \mid V_2(y^1, \bar{y}^2) \geq V_2(y^1, y^2),\ \forall\ (y^2, z^2) \in \widetilde{S}_{\alpha_2}^2\}.$$

It is clear that, if $(y^{1*}, z^{1*}) \in BR_1(y^{2*})$ and $(y^{2*}, z^{2*}) \in BR_2(y^{1*})$, $(x^{1*}, x^{2*}) = (e^{y^{1*}}, e^{y^{2*}})$ is a Nash equilibrium of the game. For computational purpose, we use the best response algorithm as outlined below:

Algorithm 1 (Best response algorithm) **Step-1** *Select initial feasible point* $(y^{2(0)}, z^{2(0)}) \in \widetilde{S}_{\alpha_2}^2$ *for player 2. Set* $k := 0$.
Step-2 *Solve convex optimization problem* $[P_1]$ *and find a point* $(y^{1(k)}, z^{1(k)}) \in BR_1(y^{2(k)})$.
Step-3 *If* $(y^{2(k)}, z^{2(k)}) \in BR_1(y^{1(k)})$, *then set* $(x^{1*}, x^{2*}) = (e^{y^{1(k)}}, e^{y^{2(k)}})$ *and stop. Otherwise, solve convex optimization problem* $[P_2]$ *and find a point* $(y^{2(k)}, z^{2(k)}) \in BR_1(y^{1(k)})$, *set* $k = k + 1$ *and go to step 2.*

If the Algorithm 1 stops, (x^{1*}, x^{2*}) is a Nash equilibrium of the game. The proof that Algorithm 1 never cycles is still an open problem.

Example 1 We consider the case of two electricity firms with four generation nodes and three distribution nodes, i.e., $N = \{1, 2, 3, 4\}$ and $M = \{1, 2, 3\}$. We take the confidence level values of player 1 and player 2 as 0.9, i.e., $\alpha_1 = \alpha_2 = 0.9$. The randomly generated data used in the model are summarized as follows:

$$\beta_{kj} = 30,\ \delta_{kj} = 1,\ c^1_{kj} = 15,\ c^2_{kj} = 12,$$
$$\varepsilon^1_{kj} = 3, \varepsilon^2_{kj} = 4, \gamma^1_{kj} = \gamma^2_{kj} = 50\ \forall\ k \in N, j \in M,$$
$$\mu^1_1 = \begin{pmatrix}0.237\\0.224\\0.240\end{pmatrix},\ \mu^1_2 = \begin{pmatrix}0.241\\0.226\\0.166\end{pmatrix},\ \mu^1_3 = \begin{pmatrix}0.159\\0.239\\0.224\end{pmatrix},\ \mu^1_4 = \begin{pmatrix}0.167\\0.165\\0.186\end{pmatrix},$$
$$\mu^2_1 = \begin{pmatrix}0.214\\0.182\\0.246\end{pmatrix},\ \mu^2_2 = \begin{pmatrix}0.218\\0.155\\0.194\end{pmatrix},\ \mu^2_3 = \begin{pmatrix}0.197\\0.162\\0.211\end{pmatrix},\ \mu^2_4 = \begin{pmatrix}0.212\\0.195\\0.171\end{pmatrix},$$
$$\Sigma^1_1 = \begin{pmatrix}0.074 & 0.056 & 0.034\\0.056 & 0.058 & 0.022\\0.034 & 0.022 & 0.035\end{pmatrix},\ \Sigma^1_2 = \begin{pmatrix}0.024 & 0.007 & 0.017\\0.007 & 0.009 & 0.006\\0.017 & 0.006 & 0.021\end{pmatrix},$$

$$\Sigma^1_3 = \begin{pmatrix}0.01 & 0.016 & 0.010\\0.016 & 0.065 & 0.038\\0.010 & 0.038 & 0.037\end{pmatrix},\ \Sigma^1_4 = \begin{pmatrix}0.042 & 0.035 & 0.040\\0.035 & 0.070 & 0.046\\0.040 & 0.046 & 0.058\end{pmatrix},$$
$$\Sigma^2_1 = \begin{pmatrix}0.043 & 0.026 & 0.030\\0.026 & 0.038 & 0.025\\0.030 & 0.025 & 0.036\end{pmatrix},\ \Sigma^2_2 = \begin{pmatrix}0.045 & 0.037 & 0.022\\0.037 & 0.052 & 0.014\\0.022 & 0.014 & 0.023\end{pmatrix},$$
$$\Sigma^2_3 = \begin{pmatrix}0.048 & 0.029 & 0.039\\0.029 & 0.026 & 0.027\\0.039 & 0.027 & 0.044\end{pmatrix},\ \Sigma^2_4 = \begin{pmatrix}0.006 & 0.002 & 0.003\\0.002 & 0.015 & 0.014\\0.003 & 0.014 & 0.034\end{pmatrix},$$
$$b^1 = \begin{pmatrix}22.138\\21.519\\20.323\\20.921\end{pmatrix},\ b^2 = \begin{pmatrix}23.277\\21.393\\22.65\\24.968\end{pmatrix}.$$

We implement Algorithm 1 on Intel® Core™ 64-bit i5-7200U CPU @ 2.50GHz × 4 and 11.6 GiB RAM on Ubuntu 18.04.2 LTS. We use **sqp** in GNU Octave 5.2.0 to solve the best response convex optimization problem of both the players. For the above data sets, Algorithm 1 converges to a point $(y^{1*}, z^{1*}, y^{2*}, z^{2*})$ such that $(y^{1*}, z^{1*}) \in BR_1(y^{2*})$ and $(y^{2*}, z^{2*}) \in BR_2(y^{1*})$. Therefore, $(x^{1*}, x^{2*}) = (e^{y^{1*}}, e^{y^{2*}})$ is a Nash equilibrium of the game and it is given by

$$x^{1*} = ((4.34, 4.3, 4.32), (4.4, 4.38, 4.25), (4.39, 4.28, 4.38), (4.04, 4.10, 4.14)),$$
$$x^{2*} = ((6.33, 6.4, 6.36), (6.19, 6.24, 6.49), (6.23, 6.44, 6.25), (6.91, 6.81, 6.71)).$$

Figure 1 shows that Algorithm 1 converges to a Nash equilibrium payoffs of both the firms after few iterations. The total CPU time to compute Nash equilibrium is 7.71 seconds.

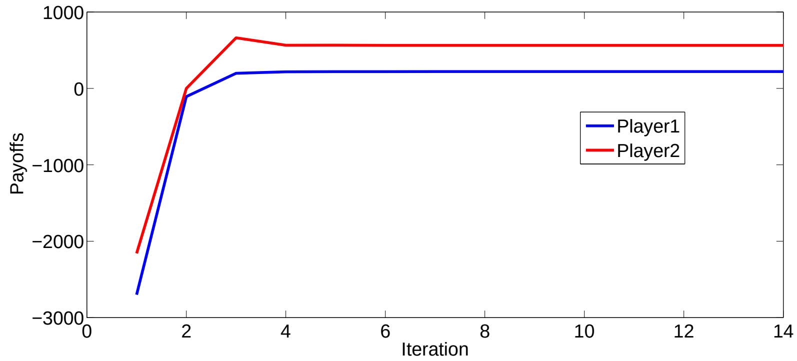

Fig. 1: Convergence of Nash equilibrium payoff

We also perform numerical experiments for a relatively large size model with 10 generation nodes and 12 distribution nodes. For this model, we take $\delta_{kj} = 1$, $\beta_{kj} = \textbf{randi}([30, 50])$, $c^1_{kj} = \textbf{randi}([12, 18])$ and $c^2_{kj} = \textbf{randi}([9, 15])$ for all $k \in N$ and $j \in M$, where $\textbf{randi}([a, b])$ generates a random integer number between closed interval $[a, b]$. For each $k \in N$ and $j \in M$, we choose ε^1_{kj} and ε^2_{kj} such that condition (33) is satisfied, and $\gamma^1_{kj} = \gamma^2_{kj} = 50$. For $k \in N$ and $i = 1, 2$, we take the mean vector $\mu^i_k = 0.15 + \frac{1}{10}\textbf{rand}(12, 1)$ and the covariance matrix $\Sigma^i_k = BB^T + s \cdot I_{12\times12}$, where $B = \frac{1}{5}\textbf{rand}(12, 12)$ and $s > 0$. The $\textbf{rand}(12,\ 1)$ generates a 12×1 random vector whose entries are between 0 and 1 and $\textbf{rand}(12,\ 12)$ generates a 12×12 random matrix whose entries are between 0 and 1. We consider 20 different instances of this model, and for each instance Algorithm 1 converges to a Nash equilibrium. The average time to compute a Nash equilibrium is 301.81 seconds. The Figure 2 represents the variation in the time for different instances.

5 Conclusion

We studied Nash equilibirum problem for the games with joint chance constraints, where the row vectors defining the random constraints are independent. We studied these games under distributionally robust framework. We considered a density based uncertainty set and four different types of two-moments based uncertainty sets. One of the moments based uncertainty sets is based on nonnegative support. For moments based uncertainty sets, we proposed a new convex reformulation of a distributionally robust joint chance constraint using logarithmic transformation. Under standard assumptions on

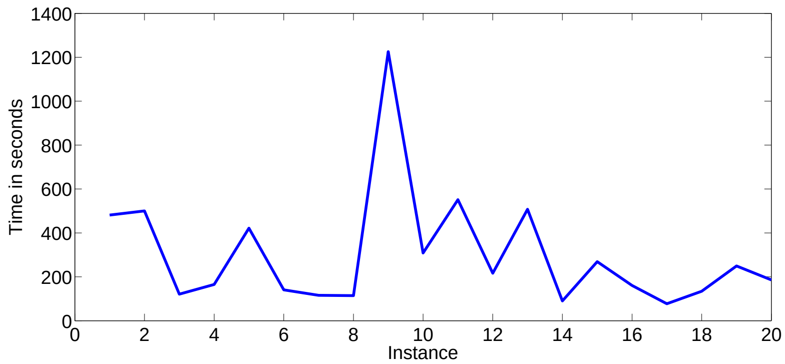

Fig. 2: Time vs Instances

players' payoff functions we showed that there exists a Nash equilibrium of a DRCCG. As an application of these games, we proposed a Cournot competition model in electricity market, which covers the generation and distribution of electricity. The best response algorithm is used to compute the Nash equilibria of various randomly generated instances of the game.

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