EQUIVALENT SECOND ORDER CONE PROGRAMS FOR DISTRIBUTION-ALLY ROBUST ZERO-SUM GAMES

Vikram Vikas, Ayush Agarwal, Navnit Yadav, Abdel Lisser

Applied Probability Trust (9 January 2021)

EQUIVALENT SECOND ORDER CONE PROGRAMS FOR DISTRIBUTIONALLY ROBUST ZERO-SUM GAMES

VIKAS VIKRAM SINGH,* *Indian Institute of Technology Delhi*

AYUSH AGARWAL,** *Indian Institute of Technology Delhi*

NAVNIT YADAV,*** *Indian Institute of Technology Delhi*

ABDEL LISSER,**** *Université Paris Saclay*

Abstract

We consider a two-player zero-sum game with random linear constraints. The probability distributions of the random constraint vectors are partially known. The available information with respect to the distribution is based mainly on the first two moments. In this vein, we formulate the random linear constraints as distributionally robust chance constraints. We consider three different types of moments based uncertainty sets. For each uncertainty set, we show that a saddle point equilibrium of the game can be obtained from the optimal solutions of a primal-dual pair of second order cone programs. We illustrate our theoretical results on randomly generated game instances of different sizes.

Keywords: Distributionally robust chance constraints; Zero-sum game; Saddle point equilibrium; Second-order cone program.

2010 Mathematics Subject Classification: Primary 91A05; 91A10; 90C15; 90C25

Secondary

---

* Postal address: Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, 110016, India. Email address: vikassingh@iitd.ac.in

** Postal address: Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, 110016, India. Email address: ayush29march1997@gmail.com

*** Postal address: Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, 110016, India. Email address: navnitydv@gmail.com

**** Postal address: Laboratoy of Signals and Systems, CentraleSupelec, Bat Breguet, 3 Rue Joliot Curie, 91190 Gif-sur-Yvette, France. Email address: abdel.lisser@l2s.centralesupelec.fr

1. Introduction

In a two-player zero-sum game, the gain of one player is the loss of the other player. We typically represent a zero-sum game with a payoff matrix where the rows and the columns are the actions of player 1 and player 2, respectively. von Neumann [34] showed that there exists a mixed strategy saddle point equilibrium (SPE) of a zero-sum game. Dantzig [14] showed the equivalence of SPE and the solution of a primal-dual pair of linear programs. Adler [1] studied the equivalence between linear programming problems and zero-sum games. Charnes [7] generalized the zero-sum game considered in [14, 34] by introducing linear inequality constraints on the mixed strategies of both the players. He showed the equivalence of an SPE of a constrained zero-sum game and an optimal solution of a primal-dual pair of linear programs. Nash [26] introduced the equilibrium concept for a finite number of rational players where each player has finite number of actions. He showed that there exists a mixed strategy Nash equilibrium for finite strategic games. Since then, general strategic games have been extensively studied in the literature [2, 15, 18]. The games discussed in the above-mentioned papers have deterministic strategy set and payoff function for each player. However, the decision making process is fed by input parameters which are usually subject to uncertainties [13, 25]. The model uncertainties can be accounted for the expected value approach in case of risk neutral decision makers. Ravat and Shanbhag [28] considered the stochastic Nash games where each player optimizes his expected value payoff subject to his expected value constraints. For risk averse players, the payoff criterion with the risk measure CVaR [22, 28] and the variance was considered in the literature [12]. Singh et al. [30, 31, 32] introduced chance-constrained games by considering a risk averse payoff criterion based on the chance constraint programming for finite strategic games with random payoffs. As for the elliptically distributed random payoffs, the authors showed the existence of a Nash equilibrium for a chance-constrained game [30], and proposed an equivalent mathematical program to compute the Nash equilibria of the game [32]. In [31], the authors considered the games where the probability distributions of the players' payoffs are partially known, and belong to given distributional uncertainty sets. They defined each player's payoff function via a distributionally robust chance constraint, and showed the existence of a Nash equilibrium. Moreover, they proposed an equivalent mathematical program based method to come-up with Nash equilibria. There are some zero-sum chance-constrained games studied in past literature [3, 6, 8, 10].

The chance-constrained games in the above-mentioned papers considered the case where players' payoffs are random variables and the strategy sets are deterministic in nature. The chance constraint based strategy sets are often considered in various applications, e.g., resource constraints in stochastic shortest path problem [11] and risk constraints in portfolio optimization [21] can be modelled using chance constraints. To the best of our knowledge, the research on the games with chance constraint based strategy set is very scarce [27, 33]. Peng et al. [27] considered an n-player general sum game with joint chance constraint for each player. They showed the existence of a Nash equilibrium when the random constraint vectors are independent, and follow a multivariate normal distribution. Singh and Lisser [33] considered a stochastic version of a two-player constrained zero-sum game studied in [7] where each player has individual chance constraints. They showed the equivalence of an SPE and an optimal solution of a primal-dual pair of a second order cone programs (SOCPs) when the random constraint vectors follow a multivariate elliptically symmetric distribution. Unlike stochastic two-player constrained zero-sum game [33], the only information we have on the probability distribution of a random constraint vector is that it belongs to a certain uncertainty set. In this paper, we consider three different types of well-known uncertainty sets [9, 16, 17] based on the first and second order moments of the random constraint vectors. For each type of uncertainty set, we show that there exists an SPE and it can be obtained from the optimal solutions of a primal-dual pair of SOCPs. We perform the numerical experiments by considering randomly generated games of different sizes. We use CVX package in MATLAB for solving equivalent SOCPs.

The structure of the rest of the paper is as follows. Section 2 contains the definition of a zero-sum game with distributionally robust chance constraints. Section 3 presents the reformulation of distributionally robust chance constraints as second order cone constraints under three different uncertainty sets. Section 4 outlines a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game. We present our numerical results in Section 5, and conclude the paper in Section 6.

2. The Model

We consider a two-player zero-sum game defined by an $n_1 \times n_2$ matrix G, where n_1 and n_2 denote the number of pure strategies of player 1 and player 2, respectively. The set of mixed strategies of player i, $i = 1,2$, is given by $X^i = \left\{ x^i \in \mathbb{R}^{n_i} \,|\, \sum_{j=1}^{n_i} x^i_j = 1, x^i_j \geq 0, \forall\, j = 1,2,\ldots,n_i \right\}$. Each component of the matrix G corresponds to the payoff for player 1 and the cost for player 2. For a given strategy pair $(x^1,x^2) \in X^1 \times X^2$, $(x^1)^T G x^2$ represents the payoff of player 1 and the cost of player 2; T denotes the transposition. For a given $x^1 \in X^1$ (resp., $x^2 \in X^2$), player 2 (resp., player 1) chooses an optimal strategy by minimizing (resp., maximizing) $(x^1)^T G x^2$ over all $x^2 \in X^2$ (resp., $x^1 \in X^1$). A strategy pair (x^1,x^2) is an SPE of the zero-sum game if x^1 (resp., x^2) is an optimal strategy of player 1 (resp., player 2) for a fixed strategy x^2 (resp., x^1) of player 2 (resp., player 1). An SPE of a zero-sum game exists in mixed strategies [34], and can be computed via a primal-dual pair of linear programs [14]. In certain cases, the players' mixed strategies are further restricted by linear inequalities. For example, in a portfolio management problem, the fractions of total amount invested in different assets incur random losses and an investor wants the total loss below certain threshold [21]. In game theoretic context these problems can be represented as a zero-sum game where nature can be considered as the adversary player. Let the mixed strategies of player 1 satisfy the following linear constraints

$$A^1 x^1 \leq b^1, \tag{1}$$

whilst the mixed strategies of player 2 satisfy the linear constraints given by

$$A^2 x^2 \geq b^2, \tag{2}$$

where A^1 is a $p \times n_1$ matrix and A^2 is a $q \times n_2$ matrix. We denote $A^1 = [a^1_1, a^1_2, \ldots, a^1_p]^T$ and $A^2 = [a^2_1, a^2_2, \ldots, a^2_q]^T$, where a^i_k represents kth row of the matrix A^i. Let $\mathcal{I}_1 = \{1,2,\ldots,p\}$ and $\mathcal{I}_2 = \{1,2,\ldots,q\}$ be the index sets for the constraints of player 1 and player 2, respectively.

Charnes [7] considered the case where A^1 and A^2 are deterministic matrices and called it a constrained zero-sum game. He showed that an SPE of a two-player constrained zero-sum game can be obtained from the optimal solutions of a primal-dual pair of linear programs. Singh and Lisser [33] considered the case when A^1 and A^2 are random matrices where each row vector follows a multivariate elliptically symmetric distribution. They formulated each random linear constraint from (1) and (2) as a chance constraint. Such games are called zero-sum chance-constrained games. They showed that an SPE of a zero-sum chance-constrained game can be obtained from the optimal solutions of a primal-dual pair of SOCPs. In this paper, we consider the case where we have no distributional knowledge of the probability distributions of the random vectors of A^1 and A^2 except their first two moments which leads to optimize over an uncertainty set. When considering the worst case scenario, we use distributionally robust framework to formulate the random linear constraints (1) and (2). The distributionally robust chance constraints of player 1 are given by

$$\inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P}\left(a_k^1 x \leq b_k^1\right) \geq \alpha_k^1,\ \forall\, k \in \mathcal{I}_1, \tag{3}$$

where F_k^1 is a probability distribution of a_k^1, $\mathcal{D}_k^1$ is the uncertainty set corresponding to the probability distribution of random vector a_k^1, and α_k^1 is the confidence level of player 1 for the kth constraint. Similarly, the distributionally robust chance constraints of player 2 are given by

$$\inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P}\left(-a_l^2 y \leq -b_l^2\right) \geq \alpha_l^2,\ \forall\, l \in \mathcal{I}_2, \tag{4}$$

where F_l^2, $\mathcal{D}_l^2$ and α_l^2 are analogously defined. Therefore, for a given $\alpha^1 = (\alpha_k^1)_{k\in\mathcal{I}_1}$ and $\alpha^2 = (\alpha_l^2)_{l\in\mathcal{I}_2}$ the feasible strategy sets of player 1 and player 2 are given by

$$S^1_{\alpha^1} = \Big\{x^1 \in X^1 |\ \inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} \geq \alpha_k^1,\ \forall\, k \in \mathcal{I}_1\Big\}, \tag{5}$$

and

$$S^2_{\alpha^2} = \Big\{x^2 \in X^2 |\ \inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P}\{-a_l^2 x^2 \leq -b_l^2\} \geq \alpha_l^2,\ \forall\, l \in \mathcal{I}_2\Big\}. \tag{6}$$

We call the matrix game G with the strategy set $S^1_{\alpha^1}$ for player 1 and the strategy set $S^2_{\alpha^2}$ for player 2 as a distributionally robust zero-sum chance-constrained game. We denote this game by Z_α. A strategy pair $(x^{1*}, x^{2*}) \in S^1_{\alpha^1} \times S^2_{\alpha^2}$ is called an SPE of the game Z_α at $\alpha = (\alpha^1, \alpha^2) \in [0,1]^p \times [0,1]^q$, if

$$(x^1)^T G x^{2*} \leq (x^{1*})^T G x^{2*} \leq (x^{1*})^T G x^2,$$

for all $x^1 \in S^1_{\alpha^1}$, $x^2 \in S^2_{\alpha^2}$.

3. Reformulation of distributionally robust chance constraints

We consider three different uncertainty sets based on the full/partial information about the mean vectors and covariance matrices of the random vectors a_k^1, $k \in \mathcal{I}_1$, and a_l^2, $l \in \mathcal{I}_2$. For each uncertainty set, chance constraints (3) and (4) are reformulated as second order cone constraints.

3.1. Moments based uncertainty sets

First, we consider the case where the first two moments are known. The uncertainty set of player i, $i = 1,2$, defined by the mean vector μ_k^i and the covariance matrix Σ_k^i of $(a_k^i)^T$, is given by

$$\mathcal{D}_k^i\left(\mu_k^i, \Sigma_k^i\right) = \left\{ F_k^i \;\middle|\; \begin{array}{c} E_{F_k^i}\left[(a_k^i)^T\right] = \mu_k^i \\ E_{F_k^i}\left[\left((a_k^i)^T - \mu_k^i\right)\left((a_k^i)^T - \mu_k^i\right)^T\right] = \Sigma_k^i \end{array} \right\}, \tag{7}$$

for all $k \in \mathcal{I}_i$. These uncertainty sets are considered in [5, 17]. As for the second uncertainty set, we assume that the first moment is known whilst the second moment is unknown. The uncertainty set of player i, $i = 1,2$, defined by the mean vector μ_k^i and the upper bound Σ_k^i on covariance matrix of $(a_k^i)^T$, is given by

$$\mathcal{D}_k^i\left(\mu_k^i, \Sigma_k^i\right) = \left\{ F_k^i \;\middle|\; \begin{array}{c} E_{F_k^i}\left[(a_k^i)^T\right] = \mu_k^i \\ E_{F_k^i}\left[\left((a_k^i)^T - \mu_k^i\right)\left((a_k^i)^T - \mu_k^i\right)^T\right] \preceq \Sigma_k^i \end{array} \right\}, \tag{8}$$

for all $k \in \mathcal{I}_i$. These uncertainty sets are considered in [9]. The last uncertainty set is based on unknown first two moments. The uncertainty set of player i, $i = 1,2$, where the mean vector of $(a_k^i)^T$ lies in an ellipsoid of size $\gamma_{k1}^i \geq 0$ centered at μ_k^i and the covariance matrix of $(a_k^i)^T$ lies in a positive semidefinite cone defined with a linear matrix inequality, is given by

$$\mathcal{D}_k^i(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \;\middle|\; \begin{array}{l} \left(\mathbb{E}_{F_k^i}[(a_k^i)^T] - \mu_k^i\right)^\top \left(\Sigma_k^i\right)^{-1} \left(\mathbb{E}_{F_k^i}[(a_k^i)^T] - \mu_k^i\right) \leq \gamma_{k1}^i, \\ COV_{F_k^i}[(a_k^i)^T] \preceq \gamma_{k2}^i \Sigma_k^i \end{array} \right\}, \tag{9}$$

for all $k \in \mathcal{I}_i$. $COV_{F_k^i}$ is a covariance operator under probability distribution F_k^i. These uncertainty sets are considered in [16, 23].

3.2. Second order cone constraint reformulation

It follows from Theorem 3.1 of [5] that the chance constraints (3) and (4) for uncertainty set (7) can be reformulated as

$$(\mu_k^1)^T x^1 + \sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}} \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1, \ \forall\, k \in \mathcal{I}_1,$$

and

$$-(\mu_l^2)^T x^2 + \sqrt{\frac{\alpha_l^2}{1-\alpha_l^2}} \left\| (\Sigma_l^2)^{\frac{1}{2}} x^2 \right\| \leq -b_l^2, \ \forall\, l \in \mathcal{I}_2,$$

where $\|\cdot\|$ is the Euclidean norm. The same reformulation based on Lagrangian duality follows from Theorem 1 of [17]. Based on the structure of uncertainty set (8), the constraint (3) can be written as

$$\inf_{\Sigma \in \mathcal{U}_k^1} \inf_{F_k^1 \in \mathcal{D}(\mu_k^1, \Sigma)} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} \geq \alpha_k^1,$$

where

$$\mathcal{D}(\mu_k^1, \Sigma) = \left\{ F_k^1 \,\middle|\, \mathbb{E}_{F_k^1}[(a_k^1)^T] = \mu_k^1, COV_{F_k^1}[(a_k^1)^T] = \Sigma \right\}$$

and

$$\mathcal{U}_k^1 = \left\{ \Sigma \,\middle|\, \Sigma \preceq \Sigma_k^1 \right\}.$$

The bound of one-sided Chebyshev inequality can be achieved by a two-point distribution given by equation (2) of [29]. Therefore, we have

$$\inf_{F_k^1 \in \mathcal{D}(\mu_k^1, \Sigma)} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} = \begin{cases} 1 - \frac{1}{1 + \frac{((\mu_k^1)^T x^1 - b_k^1)^2}{\left((x^1)^T \Sigma x^1\right)}}, & \text{if } (\mu_k^1)^T x^1 \leq b_k^1, \\ 0, & \text{otherwise.} \end{cases}$$

For the case $(\mu_k^1)^T x^1 > b_k^1$,

$$\inf_{F_k^1 \in \mathcal{D}(\mu_k^1, \Sigma)} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} = 0,$$

which leads to constraint (3) to be infeasible for any $\alpha_1 > 0$. Therefore, for $x^1 \in S_{\alpha_1}^1$ the condition $(\mu_k^1)^T x^1 \leq b_k^1$ always holds and the constraint (3) is equivalent to

$$\inf_{\Sigma \in \mathcal{U}_k^1} 1 - \frac{1}{1 + ((\mu_k^1)^T x^1 - b_k^1)^2 / ((x^1)^T \Sigma x^1)} \geq \alpha_k^1,$$

which can be reformulated as

$$h_k^1(x^1) \geq \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}}, \tag{10}$$

where

$$h_k^1(x^1) = \begin{cases} \min\limits_{\Sigma} \frac{b_k^1 - (\mu_k^1)^T x^1}{\sqrt{(x^1)^T \Sigma x^1}} \\ \text{s.t.}(i)\ \Sigma \preceq \Sigma_k^1. \end{cases} \tag{11}$$

From (11), it follows that $h_k^1(x^1) = \frac{b_k^1 - (\mu_k^1)^T x^1}{\sqrt{(x^1)^T \Sigma_k^1 x^1}}$. Then, from (10) the reformulation of (3) is given by

$$(\mu_k^1)^T x^1 + \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}} \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1, \ \forall\, k \in \mathcal{I}_1.$$

Similarly, the reformulation of (4) for uncertainty set (8) is given by

$$-(\mu_l^2)^T x^2 + \sqrt{\frac{\alpha_l^2}{1 - \alpha_l^2}} \left\| (\Sigma_l^2)^{\frac{1}{2}} x^2 \right\| \leq -b_l^2, \ \forall\, l \in \mathcal{I}_2.$$

This implies that the deterministic reformulations of (3) and (4) for uncertainty sets (7) and (8) are same. Based on the structure of the uncertainty set (9), the constraint (3) can be written as

$$\inf_{(\mu,\Sigma)\in\tilde{\mathcal{U}}_k^1} \inf_{F_k^1\in\mathcal{D}(\mu,\Sigma)} \mathbb{P}\{a_k^1x^1 \le b_k^1\} \ge \alpha_k^1,$$

where

$$\mathcal{D}(\mu,\Sigma) = \left\{F_k^1 \,\middle|\, \mathbb{E}_{F_k^1}[(a_k^1)^T] = \mu, COV_{F_k^1}[(a_k^1)^T] = \Sigma\right\}$$

and

$$\tilde{\mathcal{U}}_k^1 = \left\{(\mu,\Sigma) \,\middle|\, \left(\mu-\mu_k^1\right)^\top \left(\Sigma_k^1\right)^{-1}\left(\mu-\mu_k^1\right) \le \gamma_{k1}^1,\ \Sigma \preceq \gamma_{k2}^1\Sigma_k^1\right\}.$$

Using the similar arguments as in previous case, the constraint (3) is equivalent to

$$\inf_{(\mu,\Sigma)\in\tilde{\mathcal{U}}_k^1} 1 - \frac{1}{1+(\mu^Tx^1-b_k^1)^2/((x^1)^T\Sigma x^1)} \ge \alpha_k^1,$$

which can be reformulated as

$$\tilde{h}_k^1(x^1) \ge \sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}, \tag{12}$$

where

$$\tilde{h}_k^1(x^1) = \begin{cases} \min\limits_{\mu,\Sigma} \frac{b_k^1-\mu^Tx^1}{\sqrt{(x^1)^T\Sigma x^1}} \\ \text{s.t. } (i)\ \left(\mu-\mu_k^1\right)^\top \left(\Sigma_k^1\right)^{-1}\left(\mu-\mu_k^1\right) \le \gamma_{k1}^1, \\ (ii)\ \Sigma \preceq \gamma_{k2}^1\Sigma_k^1. \end{cases} \tag{13}$$

For the sake of simplicity, we separate problem (13) into two optimization problems

$$\tilde{h}_k^1(x^1) = \frac{b_k^1+v_1(x^1)}{\sqrt{v_2(x^1)}},$$

where

$$v_1(x^1) = \begin{cases} \min\limits_{\mu} -\mu^Tx^1 \\ \text{s.t. } \left(\mu-\mu_k^1\right)^\top \left(\Sigma_k^1\right)^{-1}\left(\mu-\mu_k^1\right) \le \gamma_{k1}^1, \end{cases} \tag{14}$$

$$v_2(x^1) = \begin{cases} \max\limits_{\Sigma}\ (x^1)^T\Sigma x^1 \\ \text{s.t. } \Sigma \preceq \gamma_{k2}^1\Sigma_k^1. \end{cases}$$

Let $\beta \ge 0$ be a Lagrange multiplier associated with the constraint of optimization problem (14). By applying the KKT conditions, the optimal solution of (14) is given by $\mu = \mu_k^1 + \frac{\sqrt{\gamma_{k1}^1}\Sigma_k^1x^1}{\sqrt{(x^1)^T\Sigma_k^1x^1}}$

and associated Lagrange multiplier is given by $\beta = \sqrt{\frac{(x^1)^T \Sigma_k^1 x^1}{4\gamma_{k1}^1}}$. Therefore, the corresponding optimal value $v_1(x^1) = -(\mu_k^1)^T x^1 - \sqrt{\gamma_{k1}^1}\sqrt{(x^1)^T \Sigma_k^1 x^1}$. Since, $u^T \Sigma u \leq u^T \gamma_{k2}^1 \Sigma_k^1 u$ for any $u \in \mathbb{R}^n$, then, $v_2(x^1) = \gamma_{k2}^1 (x^1)^T \Sigma_k^1 x^1$. Therefore, using (12) we have the following reformulation of (3)

$$(\mu_k^1)^T x^1 + \left(\sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}\sqrt{\gamma_{k2}^1} + \sqrt{\gamma_{k1}^1}\right) \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1, \tag{15}$$

for all $k \in \mathcal{I}_1$. Similarly, the reformulation of (4) is given by

$$-(\mu_l^2)^T x^2 + \left(\sqrt{\frac{\alpha_l^2}{1-\alpha_l^2}}\sqrt{\gamma_{l2}^2} + \sqrt{\gamma_{l1}^2}\right) \left\| (\Sigma_l^2)^{\frac{1}{2}} x^2 \right\| \leq -b_l^2, \tag{16}$$

for all $l \in \mathcal{I}_2$.

The reformulation of feasible strategy sets (5) and (6) for uncertainty sets (7), (8), and (9) can be written as

$$S_{\alpha^1}^1 = \left\{ x^1 \in X^1 \mid (\mu_k^1)^T x^1 + \kappa_{\alpha_k^1} \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1,\ k \in \mathcal{I}_1 \right\}, \tag{17}$$

and

$$S_{\alpha^2}^2 = \left\{ x^2 \in X^2 \mid -(\mu_l^2)^T x^2 + \kappa_{\alpha_l^2} \left\| (\Sigma_l^2)^{\frac{1}{2}} x^2 \right\| \leq -b_l^2,\ l \in \mathcal{I}_2 \right\}, \tag{18}$$

where $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$, $i = 1,2$, represents the reformulation under uncertainty sets (7) and (8), and $\kappa_{\alpha_k^i} = \left(\sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}\sqrt{\gamma_{k2}^i} + \sqrt{\gamma_{k1}^i}\right)$, $i = 1,2$, represents the reformulation for uncertainty set (9).

Proposition 1. *Consider the random constraint vectors a_k^i, $k \in \mathcal{I}_i$, $i = 1,2$, whose probability distributions belong to uncertainty sets defined by* (7)*,* (8)*,* (9)*. Then, the deterministic reformulations of* (3) *and* (4) *are given by* (17) *and* (18)*, respectively.*

We assume that the strategy sets (17) and (18) satisfy the strict feasibility condition given by Assumption 1.

Assumption 1.
1. *There exists an $x^1 \in S_{\alpha^1}^1$ such that the inequality constraints of $S_{\alpha^1}^1$ defined by* (17) *are strictly satisfied.*
2. *There exists an $x^2 \in S_{\alpha^2}^2$ such that the inequality constraints of $S_{\alpha^2}^2$ defined by* (18) *are strictly satisfied.*

The conditions given in Assumption 1 are Slater's condition which are sufficient for strong duality in a convex optimization problem. We use these conditions in order to derive equivalent SOCPs for the zero-sum game Z_α.

4. Existence and characterization of saddle point equilibrium

In this section, we show that there exists an SPE of the game Z_α if the probability distributions of the random constraint vectors of both the players belong to the uncertainty sets defined in Section 3.1. We further propose a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game Z_α.

Theorem 1. *Consider the game Z_α where the probability distributions of the row vectors a_k^i, $k \in \mathfrak{I}_i$, $i = 1,2$, belong to the uncertainty sets described in Section 3.1. Then, there exists an SPE of the game for all $\alpha \in (0,1)^p \times (0,1)^q$.*

Proof. Let $\alpha \in (0,1)^p \times (0,1)^q$. For uncertainty sets described in Section 3.1, the strategy sets $S^1_{\alpha^1}$ and $S^2_{\alpha^2}$ are given by (17) and (18), respectively. It is easy to see that $S^1_{\alpha^1}$ and $S^2_{\alpha^2}$ are convex and compact sets. The function $(x^1)^T Gx^2$ is a bilinear and continuous function. Hence, there exists an SPE from the minimax theorem of Neumann [34]. □

4.1. Equivalent primal-dual pair of second order cone programs

From the minimax theorem [34], (x^{1*}, x^{2*}) is an SPE for the game Z_α if and only if

$$x^{1*} \in \arg\max_{x^1 \in S^1_{\alpha^1}} \min_{x^2 \in S^2_{\alpha^2}} \left(x^1\right)^T Gx^2, \tag{19}$$

$$x^{2*} \in \arg\min_{x^2 \in S^2_{\alpha^2}} \max_{x^1 \in S^1_{\alpha^1}} \left(x^1\right)^T Gx^2. \tag{20}$$

We start with problem $\min_{x^2 \in S^2_{\alpha^2}} \max_{x^1 \in S^1_{\alpha^1}} \left(x^1\right)^T Gx^2$. The inner optimization problem $\max_{x^1 \in S^1_{\alpha^1}} \left(x^1\right)^T Gx^2$ can be equivalently written as

$$\left.\begin{array}{l}
\max\limits_{x^1,\left(t_k^1\right)_{k\in\mathcal{I}_1}} \left(x^1\right)^T G x^2 \\
\text{s.t.} \\
\quad (i)\ -\left(x^1\right)^T \mu_k^1 - \kappa_{\alpha_k^1} \left\|t_k^1\right\| + b_k^1 \geq 0,\ \forall\, k\in\mathcal{I}_1 \\
\quad (ii)\ t_k^1 - \left(\Sigma_k^1\right)^{\frac{1}{2}} x^1 = 0, \quad \forall \quad k\in\mathcal{I}_1 \\
\quad (iii)\ \sum\limits_{j=1}^{n_1} x_j^1 = 1, \\
\quad (iv)\ x_j^1 \geq 0, \quad \forall \quad j=1,2,\ldots,n_1.
\end{array}\right\} \tag{21}$$

Let $\lambda^1 = \left(\lambda_k^1\right)_{k\in\mathcal{I}_1} \in \mathbb{R}^p$, $\delta_k^1 \in \mathbb{R}^{n_1}$, $k\in\mathcal{I}_1$, and $\nu^1 \in \mathbb{R}$ be the Lagrange multipliers of constraints $(i), (ii)$, and (iii) of (21), respectively. Then, the Lagrangian dual problem of the SOCP (21) is an SOCP [4, 24]. Moreover, the duality gap is zero according to Assumption 1 . Therefore, the problem $\min_{x^2\in S^2_{\alpha^2}} \max_{x^1\in S^1_{\alpha^1}} \left(x^1\right)^T G x^2$ is equivalent to the following SOCP

$$\left.\begin{array}{l}
\min\limits_{x^2,\nu^1,(\delta_k^1)_{k\in\mathcal{I}_1},(\lambda_k^1)_{k\in\mathcal{I}_1}} \nu^1 + \sum\limits_{k\in\mathcal{I}_1} \lambda_k^1 b_k^1 \\
\text{s.t.} \\
\quad (i)\ Gx^2 - \sum\limits_{k\in\mathcal{I}_1} \lambda_k^1 \mu_k^1 - \sum\limits_{k\in\mathcal{I}_1} (\Sigma_k^1)^{\frac{1}{2}} \delta_k^1 \leq \nu^1 \mathbf{1}_{n_1}, \\
\quad (ii)\ -(\mu_l^2)^T x^2 + \kappa_{\alpha_l^2} \left\|(\Sigma_l^2)^{\frac{1}{2}} x^2\right\| \leq -b_l^2, \quad \forall \quad l\in\mathcal{I}_2, \\
\quad (iii)\ ||\delta_k^1|| \leq \lambda_k^1 \kappa_{\alpha_k^1}, \forall\, k\in\mathcal{I}_1, \\
\quad (iv)\ \sum\limits_{j=1}^{n_2} x_j^2 = 1, \\
\quad (v)\ x_j^2 \geq 0,\ \forall\ j=1,2,\ldots,n_2 \\
\quad (vi)\ \lambda_k^1 \geq 0,\ \forall\ k\in\mathcal{I}_1,
\end{array}\right\} \tag{P}$$

where $\mathbf{1}_{n_1}$ is an $n_1 \times 1$ vector of ones. Similarly, problem $\max_{x^1 \in S^1_{\alpha^1}} \min_{x^2 \in S^2_{\alpha^2}} \left(x^1\right)^T G x^2$ is equivalent to the following SOCP

$$\left.\begin{aligned}
&\max_{x^1,\nu^2,(\delta_l^2)_{l\in\mathfrak{I}_2},(\lambda_l^2)_{l\in\mathfrak{I}_2}} \quad \nu^2 + \sum_{l\in\mathfrak{I}_2} \lambda_l^2 b_l^2 \\
&\text{s.t.} \\
&\quad (i)\ G^T x^1 - \sum_{l\in\mathfrak{I}_2} \lambda_l^2 \mu_l^2 - \sum_{l\in\mathfrak{I}_2} (\Sigma_l^2)^{\frac{1}{2}} \delta_l^2 \geq \nu^2 \mathbf{1}_{n_2}, \\
&\quad (ii)\ (\mu_k^1)^T x^1 + \kappa_{\alpha_k^1} \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1, \forall\, k \in \mathfrak{I}_1, \\
&\quad (iii)\ ||\delta_l^2|| \leq \lambda_l^2 \kappa_{\alpha_l^2},\ \forall\, l \in \mathfrak{I}_2, \\
&\quad (iv)\ \sum_{j=1}^{n_1} x_j^1 = 1, \\
&\quad (v)\ x_j^1 \geq 0,\ \forall\, j = 1,2,\ldots,n_1, \\
&\quad (vi)\ \lambda_l^2 \geq 0,\ \forall\, l \in \mathfrak{I}_2.
\end{aligned}\right\} \tag{D}$$

It follows from the duality theory of second order cone programming problem that (P) and (D) form a primal-dual pair [4, 24].

Remark 1. For $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$, $i = 1,2$, (P) and (D) represent the primal-dual pair of SOCPs for the uncertainty sets defined by (7) and (8). For $\kappa_{\alpha_k^i} = \left(\sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}\sqrt{\gamma_{k2}^i} + \sqrt{\gamma_{k1}^i}\right)$, $i = 1,2$, (P) and (D) represent the primal-dual pair of SOCPs for the uncertainty set defined by (9).

Next, we show that the equivalence between the optimal solutions of (P)-(D) and an SPE of the game Z_α.

Theorem 2. *Consider the game Z_α where the probability distributions of the random constraint vectors a_k^i, $k \in \mathfrak{I}_i$, $i = 1,2$, belong to the uncertainty sets defined by* (7)*,* (8)*,* (9)*. Let Assumption 1 holds. Then, for a given $\alpha \in (0,1)^p \times (0,1)^q$, (x^{1*}, x^{2*}) is an SPE of the game Z_α if and only if there exists $(\nu^{1*}, (\delta_k^{1*})_{k\in\mathfrak{I}_1}, \lambda^{1*})$ and $(\nu^{2*}, (\delta_l^{2*})_{l\in\mathfrak{I}_2}, \lambda^{2*})$ such that $(x^{2*}, \nu^{1*}, (\delta_k^{1*})_{k\in\mathfrak{I}_1}, \lambda^{1*})$ and $(x^{1*}, \nu^{2*}, (\delta_l^{2*})_{l\in\mathfrak{I}_2}, \lambda^{2*})$ are optimal solutions of* (P) *and* (D)*, respectively.*

Proof. Let (x^{1*}, x^{2*}) be an SPE of the game Z_α. Then, x^{1*} and x^{2*} are the solutions of (19) and (20) respectively. Therefore, there exists $(\nu^{1*}, (\delta_k^{1*})_{k\in\mathfrak{I}_1}, \lambda^{1*})$ and $(\nu^{2*}, (\delta_l^{2*})_{l\in\mathfrak{I}_2}, \lambda^{2*})$ such that $(x^{2*}, \nu^{1*}, (\delta_k^{1*})_{k\in\mathfrak{I}_1}, \lambda^{1*})$ and $(x^{1*}, \nu^{2*}, (\delta_l^{2*})_{l\in\mathfrak{I}_2}, \lambda^{2*})$ are optimal solutions of (P) and (D) respectively.

Let $(x^{2*}, \nu^{1*}, (\delta_k^{1*})_{k\in\mathfrak{I}_1}, \lambda^{1*})$ and $(x^{1*}, \nu^{2*}, (\delta_l^{2*})_{l\in\mathfrak{I}_2}, \lambda^{2*})$ be optimal solutions of (P) and (D) respectively. Under Assumption 1, (P) and (D) are strictly feasible. Therefore, strong duality holds for primal-dual pair (P)-(D). Then, we have

$$\nu^{1*} + \sum_{k\in\mathfrak{I}_1} \lambda_k^{1*} b_k^1 = \nu^{2*} + \sum_{l\in\mathfrak{I}_2} \lambda_l^{2*} b_l^2. \tag{22}$$

Consider the constraint (i) of (P) at optimal solution $(x^{2*}, \nu^{1*}, (\delta_k^{1*})_{k\in\mathfrak{I}_1}, \lambda^{1*})$ and multiply it by $(x^1)^T$, where $x^1 \in S^1_{\alpha_1}$. Then, by using Cauchy-Schwartz inequality, we have

$$(x^1)^T G x^{2*} \leq \nu^{1*} + \sum_{k\in\mathfrak{I}_1} \lambda_k^{1*} b_k^1, \; \forall\, x^1 \in S^1_{\alpha_1}. \tag{23}$$

Similarly, we have

$$(x^{1*})^T G x^2 \geq \nu^{2*} + \sum_{l\in\mathfrak{I}_2} \lambda_l^{2*} b_l^2, \; \forall\, x^2 \in S^2_{\alpha_2}. \tag{24}$$

Take $x^1 = x^{1*}$ and $x^2 = x^{2*}$ in (23) and (24), then from (22), we get

$$(x^{1*})^T G x^{2*} = \nu^{1*} + \sum_{k\in\mathfrak{I}_1} \lambda_k^{1*} b_k^1 = \nu^{2*} + \sum_{l\in\mathfrak{I}_2} \lambda_l^{2*} b_l^2. \tag{25}$$

It follows from (23), (24), and (25) that (x^{1*}, x^{2*}) is an SPE of the game Z_α. □

5. Numerical Results

For illustration purpose, we consider an instance of a zero-sum game with random constraints. We compute the saddle point equilibria of the game by solving the SOCPs (P) and (D). We use the convex programs solver CVX on MATLAB for solving the SOCPs [19, 20]. Consider a zero-sum game described by the following 4×4 payoff matrix G

$$G = \begin{pmatrix} 1 & 4 & 4 & 2 \\ 5 & 4 & 4 & 2 \\ 3 & 5 & 4 & 3 \\ 3 & 2 & 3 & 1 \end{pmatrix}.$$

We consider the stochastic linear constraints defined by 3×4 random matrices A^1 and A^2. The mean vectors and covariance matrices of the row vectors of A^1 and A^2 are summarized as below

$$\mu_1^1 = \begin{pmatrix} 11 \\ 12 \\ 9 \\ 11 \end{pmatrix},\ \mu_2^1 = \begin{pmatrix} 14 \\ 14 \\ 15 \\ 12 \end{pmatrix},\ \mu_3^1 = \begin{pmatrix} 10 \\ 11 \\ 19 \\ 19 \end{pmatrix}, \mu_1^2 = \begin{pmatrix} 19 \\ 17 \\ 18 \\ 11 \end{pmatrix},$$

$$\mu_2^2 = \begin{pmatrix} 18 \\ 18 \\ 14 \\ 19 \end{pmatrix}, \mu_3^2 = \begin{pmatrix} 15 \\ 14 \\ 19 \\ 9 \end{pmatrix},\ b^1 = \begin{pmatrix} 24 \\ 24 \\ 24 \end{pmatrix},\ b^2 = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}.$$

$$\Sigma_1^1 = \begin{pmatrix} 12 & 3 & 3 & 3 \\ 3 & 10 & 2 & 4 \\ 3 & 2 & 12 & 2 \\ 3 & 4 & 2 & 10 \end{pmatrix}, \Sigma_2^1 = \begin{pmatrix} 10 & 2 & 2 & 3 \\ 2 & 10 & 4 & 4 \\ 2 & 4 & 10 & 2 \\ 3 & 4 & 2 & 10 \end{pmatrix},$$

$$\Sigma_3^1 = \begin{pmatrix} 10 & 3 & 3 & 3 \\ 3 & 12 & 2 & 3 \\ 3 & 2 & 12 & 2 \\ 3 & 3 & 2 & 10 \end{pmatrix}, \Sigma_1^2 = \begin{pmatrix} 12 & 3 & 3 & 3 \\ 3 & 12 & 3 & 3 \\ 3 & 3 & 10 & 3 \\ 3 & 3 & 3 & 10 \end{pmatrix},$$

$$\Sigma_2^2 = \begin{pmatrix} 12 & 3 & 2 & 3 \\ 3 & 10 & 4 & 2 \\ 2 & 4 & 12 & 2 \\ 3 & 2 & 2 & 12 \end{pmatrix}, \Sigma_3^2 = \begin{pmatrix} 12 & 2 & 3 & 3 \\ 2 & 10 & 2 & 3 \\ 3 & 2 & 10 & 3 \\ 3 & 3 & 3 & 10 \end{pmatrix}.$$

Table 1 summarizes the saddle point equilibria of the game Z_α for various values of α for all three uncertainty sets defined in Section 3.1.

We also perform numerical experiments by considering various random instances of the game with different sizes. We generate the data using the integer random number generator **randi**. We take A=**randi**(10, m, n). It generates an $m \times n$ integer matrix whose entries are not more than 10. We take mean vectors corresponding to the constraints of player 1 as μ_k^1=**randi**$\big([10m, 12m], m, 1\big)$, $k \in \mathcal{I}_1$. It generates an $m \times 1$ integer vector whose entries are within interval $[10m, 12m]$. We take the mean vectors corresponding to the constraints of player 2 as μ_l^2=**randi**$(n, n, 1)$, $l \in \mathcal{I}_2$. We generate the covariance matrices $\{\Sigma_k^1\}_{k=1}^p$ and

TABLE 1: Saddle point equilibrium for different uncertainty sets

α		Saddle Point Equilibrium		Value of the game	Uncertainty set
α^1	α^2	x^{1*}	x^{2*}		
(0.9, 0.9, 0.9)	(0.9, 0.9, 0.9)	$(0,3856,0.6144,0)$	$(0.0662,0,0.3191,0.6147)$	3.13	Uncertainty set (7) or (8)
(0.95, 0.95, 0.95)	(0.95, 0.95, 0.95)	$(0.1992,0.4140,0.2978,0.0890)$	$(0.2328,0.0628,0.4275,0.2769)$	3.34	
(0.9, 0.9, 0.9)	(0.9, 0.9, 0.9)	$(0.0216,0.4609,0.5175,0)$	$(0.0638,0,0.4041,0.5321)$	3.2	Uncertainty set (9) with γ_1=0.3, γ_2=0.9
(0.95, 0.95, 0.95)	(0.95, 0.95, 0.95)	$(0.3193,0.3226,0.1728,0.1853)$	$(0.2674,0.1490,0.4109,0.1727)$	3.28	

$\{\Sigma_l^2\}_{l=1}^q$, corresponding to the constraints of player 1 and player 2 respectively, by setting $\Sigma_k^1 = Q_1 + Q_1^T + \theta_1 \cdot I_{m\times m}$ and $\Sigma_l^2 = Q_2 + Q_2^T + \theta_2 \cdot I_{n\times n}$, where Q_1=**randi**$(5,m)$ and Q_2=**randi**$(5,n)$. The matrix Q_1 is an $m\times m$ randomly generated integer matrix whose entries are not more than 5, and the matrix Q_2 is an $n\times n$ randomly generated integer matrix whose entries are not more than 5. For a given k, $I_{k\times k}$ is an $k\times k$ identity matrix. We set the parameters θ_1 and θ_2 sufficiently large so that the matrices $\{\Sigma_k^1\}_{k=1}^p$ and $\{\Sigma_l^2\}_{l=1}^q$ are positive definite. In our experiments, we take $\theta_1 = 2m$ and $\theta_2 = 2n$. We take the bounds defining the constraints of both the players as b=**randi**$([13m,14m],p,1)$ and d = **randi**$\left(\frac{n}{4},q,1\right)$. The confidence level for each chance constraint is taken as 0.95. Table 2 summarizes the average time for solving SOCPs (P) and (D).

6. Conclusions

We show the existence of a mixed strategy SPE for a two-player distributionally robust zero-sum chance-constrained game under three different uncertainty sets. The saddle point equilibria of the game can be obtained from the optimal solutions of a primal-dual pair of SOCPs. We compute the saddle point equilibria of an instance of the game Z_α by solving SOCPs (P) and (D). We perform the numerical experiments by considering randomly generated games of different sizes.

TABLE 2: Average time for solving SOCPs (P)-(D)

No. of instances	Number of actions		Number of constraints		Average time (s)	
	m	n	p	q	(P)	(D)
10	50	60	20	25	2.92	2.94
10	100	120	40	50	24.27	25.78
10	160	160	60	60	93.76	90.09

Acknowledgement

This research was supported by DST/CEFIPRA Project No. IFC/4117/DST-CNRS-5th call/2017-18/2 and CNRS Project No. AR/SB:2018-07-440.

References

[1] ADLER, I. (2013). The equivalence of linear programs and zero-sum games. *International Journal of Game Theory* **42,** 165–177.

[2] BASAR, T. AND OLSDER, G. J. (1999). *Dynamic noncooperative game theory* 2nd ed. SIAM, Philadelphia, PA.

[3] BLAU, R. A. (1974). Random-payoff two person zero-sum games. *Operations Research* **22,** 1243–1251.

[4] BOYD, S. AND VANDENBERGHE, L. (2004). *Convex Optimization*. Cambridge University Press, New York.

[5] CALAFIORE, G. C. AND GHAOUI, L. E. (2006). On distributionally robust chance-constrained linear programs. *Journal of Optimization Theory and Applications* **130,** 1–22.

[6] CASSIDY, R. G., FIELD, C. A. AND KIRBY, M. J. L. (1972). Solution of a satisficing model for random payoff games. *Management Science* **19,** 266–271.

[7] CHARNES, A. (1953). Constrained games and linear programming. *Proceedings of National Academy of Sciences of the USA* **39,** 639–641.

[8] CHARNES, A., KIRBY, M. J. L. AND RAIKE, W. M. (1968). Zero-zero chance-constrained games. *Theory of Probability and its Applications* **13,** 628–646.

[9] CHENG, J., DELAGE, E. AND LISSER, A. (2014). Distributionally robust stochastic knapsack problem. SIAM *Journal of Optimization* **24,** 1485–1506.

[10] CHENG, J., LEUNG, J. AND LISSER, A. (2016). Random-payoff two-person zero-sum game with joint chance constraints. *European Journal of Operational Research* **251,** 213–219.

[11] CHENG, J. AND LISSER, A. (2012). A second-order cone programming approach for linear programs with joint probabilistic constraints. *Operations Research Letters* **40,** 325–328.

[12] CONEJO, A. J., NOGALES, F. J., ARROYO, J. M. AND GARCÍA-BERTRAND, R. (2004). Risk-constrained self-scheduling of a thermal power producer. *IEEE Transactions on Power Systems* **19,** 1569–1574.

[13] COUCHMAN, P., KOUVARITAKIS, B., CANNON, M. AND PRASHAD, F. (2005). Gaming strategy for electric power with random demand. *IEEE Transactions on Power Systems* **20,** 1283–1292.

[14] DANTZIG, G. B. (1951). A proof of the equivalence of the programming problem and the game problem. In *Activity analysis of production and allocation*. ed. T. Koopmans. John Wiley Sons, New York pp. 330–335.

[15] DEBREU, G. (1952). A social equilibrium existence theorem. *Proceedings of National Academy of Sciences* **38,** 886–893.

[16] DELAGE, E. AND YE, Y. (2010). Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Operations Research* **58,** 595–612.

[17] EL-GHAOUI, L., OKS, M. AND OUSTRY, F. (2003). Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. *Operations Research* **51,** 543–556.

[18] FAN, K. (1966). Applications oi a theorem concerning sets with convex sections. *Mathematische Annalen* **163,** 189–203.

[19] GRANT, M. AND BOYD, S. (2008). Graph implementations for nonsmooth convex programs. In *Recent Advances in Learning and Control*. ed. V. Blondel, S. Boyd, and H. Kimura. Lecture Notes in Control and Information Sciences. Springer-Verlag Limited pp. 95–110. `http://stanford.edu/~boyd/graph$_$dcp.html`.

[20] GRANT, M. AND BOYD, S. CVX: Matlab software for disciplined convex programming, version 2.1 beta. `http://cvxr.com/cvx` March 2014.

[21] JI, R. AND LEJEUNE, M. A. (2018). Risk-budgeting multi-portfolio optimization with portfolio and marginal risk constraints. *Annals of Operations Research* **262,** 547–578.

[22] KANNAN, A., SHANBHAG, U. V. AND KIM, H. M. (2013). Addressing supply-side risk in uncertain power markets: stochastic Nash models, scalable algorithms and error analysis. *Optimization Methods and Software* **28,** 1095–1138.

[23] LIU, J., LISSER, A. AND CHEN, Z. Distributionally robust chance constrained geometric optimization 2019.

[24] LOBO, M. S., VANDENBERGHE, L., BOYD, S. AND LEBRET, H. (1998). Applications of second-order cone programming. *Linear Algebra and its Applications* **284,** 193–228.

[25] MAZADI, M., ROSEHART, W. D., ZAREIPOUR, H., MALIK, O. P. AND OLOOMI, M. (2013). Impact of wind integration on electricity markets: A chance-constrained Nash Cournot model. *International Transactions on Electrical Energy Systems* **23,** 83–96.

[26] NASH, J. F. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences* **36,** 48–49.

[27] PENG, S., SINGH, V. V. AND LISSER, A. (2018). General sum games with joint chance constraints. *Operations Research Letters* **56,** 482—486.

[28] RAVAT, U. AND SHANBHAG, U. V. (2011). On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games. *SIAM Journal of Optimization* **21,** 1168–1199.

[29] RUJEERAPAIBOON, N., KUHN, D. AND WIESEMANN, W. (2018). Chebyshev inequalities for products of random variables. *Mathematics of Operations Research* **43,** 887–918.

[30] SINGH, V. V., JOUINI, O. AND LISSER, A. (2016). Existence of Nash equilibrium for chance-constrained games. *Operations Research Letters* **44,** 640–644.

[31] SINGH, V. V., JOUINI, O. AND LISSER, A. (2017). Distributionally robust chance-constrained games: Existence and characterization of Nash equilibrium. *Optimization Letters* **11,** 1385–1405.

[32] SINGH, V. V. AND LISSER, A. (2018). A characterization of Nash equilibrium for the games with random payoffs. *Journal of Optimization Theory and Applications* **178,** 998–1013.

[33] SINGH, V. V. AND LISSER, A. (2019). A second order cone programming formulation for zero sum game with chance constraints. *European Journal of Operational Research* **275,** 839–845.

[34] VON NEUMANN, J. (1928). On the theory of games. *Math. Annalen* **100,** 295–320.