In-domain stabilization of block diagonal infinite-dimensional systems with time-varying input delays

Hugo Lhachemi, Christophe Prieur, Robert Shorten

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***Abstract*—This paper is concerned with the in-domain stabilization of a class of block diagonal infinite-dimensional systems in the presence of an uncertain and time-varying delay in the distributed control input. Two actuation schemes are considered. The first one assumes a control input that is fully distributed over the domain. The second one assumes that the control input is finite-dimensional, acting over the domain via a bounded operator. In both cases, the control design strategy consists in a predictor feedback law synthesized on a finite-dimensional truncated LTI model capturing the unstable dynamics of the original infinite-dimensional system. The predictor feedback law is designed based on the knowledge of the nominal value of the uncertain and time-varying input delay. In the second actuation scheme, the case of distinct input delays in the different scalar control input channels is considered.**

***Index Terms*—Delayed distributed actuation, partial differential equations, predictor feedback law.**

I. INTRODUCTION

Stabilization of open-loop unstable partial differential equations (PDEs) in the presence of delays, either in the control input [5], [10], [12], [13], [16], [19]–[23] or in the state [4], [6]–[9], [15], [26], is an active topic of research. In this paper, we are focused on the first class of problems. Two types of approaches have been developed for the boundary stabilization of PDEs in the presence of arbitrarily large delays in the control input. The first approach relies on backstepping transformations. Such a control design strategy was reported in [10] for the boundary feedback stabilization of an unstable reaction-diffusion equation under a constant input delay. More recently, a second approach taking advantage of the predictor feedback strategy for finite-dimensional LTI systems was reported in [22]. In this work, the predictor feedback law is designed on a finite-dimensional truncated model capturing the unstable modes of the distributed parameter system. The stability analysis of the resulting closed-loop infinite-dimensional system was carried out via a Lyapunov-based argument. The same approach was reused in [5] for the boundary stabilization of a linear Kuramoto-Sivashinsky equation under constant input delay. This approach was then generalized to a class of diagonal infinite-dimensional systems in [12], [16] for constant input delays and then in [13] for fast time-varying input delays. While the above approaches apply for boundary control inputs, very few reported works deal with the in-domain stabilization of PDEs in the presence of a long input delay. The recent work [23] tackles this problem for an unstable reaction-diffusion equation with Dirichlet boundary conditions and a constant delay in the in-domain control input. The control scheme assumes that the control input is fully distributed. The proposed control design strategy employs a backstepping transformation that leads to the kernel functions presenting singular points inducing technical challenges in the analysis.

In this work, we consider a class of block-diagonal infinite-dimensional systems presenting a finite number of unstable modes (counted with multiplicity) while the infinite-dimensional stable part of the system is assumed to be exponentially stable. The former assumption (finite number of unstable modes) is generally seen as a necessary condition to achieve the robust stabilization of the plant. In this context, our objective is to develop a control strategy for the stabilization of the aforementioned class of block-diagonal infinite-dimensional systems in the presence of uncertain and time-varying delays in the in-domain control input. This is motivated by the fact that input delays are ubiquitous in practical applications. Their occurrence induce challenges since they introduce an infinite number of modes which can provoke instabilities and hence cannot be neglected during control design [25]. Two control schemes are considered. The first one assumes a fully distributed control input, i.e. a configuration that allows to freely impose the value of the command input at any point of the domain. The second one assumes that the actual control input is finite dimensional and applies to the domain via a bounded operator. In the latter configuration, we further consider the case of distinct uncertain and time-varying input delays in the different scalar control inputs. In both configurations, the control design is performed on a finite-dimensional truncated model capturing the finite number of unstable modes of the distributed parameter system. The subsequent controller takes the form of a predictor feedback law designed based on the nominal value of the uncertain and time-varying input delay [13], [14]. We assess that this control strategy preserves the stability property of the residual infinite-dimensional dynamics, thus achieving the exponential stabilization of the closed-loop system. In essence, this control design procedure is similar to early-lumping approximation methods reported for the stabilization of autonomous infinite dimensional plants by means of a bounded input operator [18]. In this context, our main contribution is the handling of the impact of the input delays which introduce an infinite-dimensional dynamics in the loop [25] while their time-varying nature induces a non-autonomous closed-loop system. The obtained results are applied to the in-domain stabilization of a reaction-diffusion equation with Robin boundary conditions and also to the stabilization of a wave equation presenting a Kelvin-Voigt damping parameter.

This paper is organized as follows. The investigated control problem setting is introduced in Section II. The case of a fully distributed control input is discussed in Section III while the case of a finite-dimensional control input acting via a bounded operator is described in Section IV. Finally, concluding remarks are provided in Section V.

II. PROBLEM SETTING

A. Notation

The sets of non-negative integers, real, non-negative real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{C}$, respectively. The real and imaginary parts of a complex number z are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively. The field $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. The set of

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n-dimensional vectors over $\mathbb{K}$ is denoted by $\mathbb{K}^n$ and is endowed with the Euclidean norm $\|x\| = \sqrt{x^* x}$. The set of $n \times m$ matrices over $\mathbb{K}$ is denoted by $\mathbb{K}^{n \times m}$ and is endowed with the induced norm denoted by $\|\cdot\|$. For any symmetric matrix $P \in \mathbb{R}^{n \times n}$, $P \succ 0$ (resp. $P \succeq 0$) means that P is positive definite (resp. positive semi-definite). The set of symmetric positive definite matrices of order n is denoted by $\mathbb{S}_n^{+*}$. For any symmetric matrix $P \in \mathbb{R}^{n \times n}$, $\lambda_m(P)$ and $\lambda_M(P)$ denote the smallest and largest eigenvalues of P, respectively. The range of an operator in denoted by $R(\cdot)$. For any $t_0 > 0$, we say that $\varphi \in \mathcal{C}^0(\mathbb{R};\mathbb{R})$ is a *transition signal over* $[0, t_0]$ if $0 \leq \varphi \leq 1$, $\varphi|_{(-\infty,0]} = 0$, and $\varphi|_{[t_0,+\infty)} = 1$.

B. Block diagonal operators

We denote by $\mathcal{H}$ a separable Hilbert space over the field $\mathbb{R}$.

Definition 1: $(\Pi_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$, *with* $\Pi_n \neq 0$ *for all* $n \geq 1$, *is said to be a (finite or infinite) family of complete orthogonal projections if*

1) $(\Pi_n)^2 = \Pi_n$ *for all* $n \geq 1$*;*
2) $\langle \Pi_n x, y \rangle = \langle x, \Pi_n y \rangle$ *for all* $n \geq 1$ *and all* $x, y \in \mathcal{H}$*;*
3) $\Pi_n \Pi_m = 0$ *for all* $n \neq m$*;*
4) $z = \sum_{n\geq 1} \Pi_n z$ *for all* $z \in \mathcal{H}$.

We say that $(A_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$ *is compatible with the family of complete orthogonal projections* $(\Pi_n)_{n\geq 1}$ *if*

$$A_n \Pi_n = \Pi_n A_n \tag{1}$$

for all $n \geq 1$.

In the context of Definition 1, we have that $\|z\|^2 = \sum_{n\geq 1} \|\Pi_n z\|^2$ for all $z \in \mathcal{H}$. In this paper, we extensively use the following lemma which is a slight and straightforward variation of [11, Lem. 2.1].

Lemma 1: *Let* $(A_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$ *which is compatible with* $(\Pi_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$ *a family of complete orthogonal projections. We assume that there exists* $g \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$ *such that*[1]

$$\|e^{A_n t} \Pi_n\| \leq g(t) \tag{2}$$

for all $t \geq 0$ *and* $n \geq 1$*. Then* $T(t)$ *defined, for all* $t \geq 0$ *and* $z \in \mathcal{H}$*, by*

$$T(t)z = \sum_{n\geq 1} e^{A_n t} \Pi_n z \tag{3}$$

is a C_0*-semigroup on* $\mathcal{H}$ *whose infinitesimal generator is given by*

$$\mathcal{A}z = \sum_{n\geq 1} A_n \Pi_n z \tag{4}$$

for any $z \in D(\mathcal{A})$ *with*

$$D(\mathcal{A}) = \left\{ z \in \mathcal{H} \,:\, \sum_{n\geq 1} \|A_n \Pi_n z\|^2 < \infty \right\}. \tag{5}$$

Remark 1: We note that, for any given $z \in \mathcal{H}$, $T_N(t)z = \sum_{n=1}^{N} e^{A_n t} \Pi_n z$ converges uniformly in t on compact intervals to $T(t)$ as $N \to +\infty$. This property is one of the key assumptions for early-lumping methods as described in [18]. $\circ$

C. Problem setting

In this work we consider the abstract system

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + f(t) \tag{6a}$$

$$X(0) = X_0 \tag{6b}$$

where, under the assumptions of Lemma 1, $\mathcal{A}$ takes the form (4) of the infinitesimal generator of the C_0-semigroup $T(t)$ given by (3). The structure of function $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{H})$, which will depend on a delayed version of the actual control input u, will be specified later. Here $X_0 \in \mathcal{H}$ represents the initial condition. Then the mild solution $X \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})$ of (6) is given by

$$X(t) = T(t)X_0 + \int_0^t T(t-s) f(s) \mathrm{d}s.$$

We introduce $x_n(t) = \Pi_n X(t)$ and we have that $\|X(t)\|^2 = \sum_{n\geq 1} \|x_n(t)\|^2$. From (1) and as $A_n, \Pi_n \in \mathcal{L}(\mathcal{H})$, we have $e^{A_n t} \Pi_n = \Pi_n e^{A_n t}$. Then we infer from $\Pi_n \in \mathcal{L}(\mathcal{H})$ and $\Pi_n \Pi_m = 0$ for $n \neq m$ that

$$\begin{aligned} x_n(t) &= \Pi_n T(t) X_0 + \int_0^t \Pi_n T(t-s) f(s) \, \mathrm{d}s \\ &= \sum_{m\geq 1} \Pi_n \Pi_m e^{A_m t} X_0 + \int_0^t \sum_{m\geq 1} \Pi_n \Pi_m e^{A_m (t-s)} f(s) \, \mathrm{d}s \\ &= e^{A_n t} x_n(0) + \int_0^t e^{A_n (t-s)} \Pi_n f(s) \, \mathrm{d}s \end{aligned} \tag{7}$$

for all $n \geq 1$ and $t \geq 0$.

III. Fully distributed control input

In this section, we consider the case of a fully distributed control input. Specifically, we consider the abstract boundary control system (6) with $f(t) = u(t - D(t))$ where $u(t) \in \mathcal{H}$ is the control input with $u(\tau) = 0$ for $\tau \leq 0$ and $D(t) \in [D_0 - \delta, D_0 + \delta]$ is an uncertain and time-varying control input delay. Thus, the studied system takes the following form:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + u(t - D(t)) \tag{8a}$$

$$X(0) = X_0 \tag{8b}$$

A. Main result

The main result of this section is the following theorem.

Theorem 1: *Let* $D_0, t_0 > 0$ *be given. Let* $(A_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$ *be compatible with* $(\Pi_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$ *a family of complete orthogonal projections. We assume that there exists* $N_0 \geq 1$ *such that*[2]*:*

- $d_n = \dim(R(\Pi_n)) < \infty$ *for all* $1 \leq n \leq N_0$*;*
- *there exist* $M \geq 1$ *and* $\sigma > 0$ *such that* $\|e^{A_n t} \Pi_n\| \leq M e^{-\sigma t}$ *for all* $n \geq N_0 + 1$ *and* $t \geq 0$.

We denote by $T(t)$ *the* C_0*-semigroup defined by (3) and we consider* $\mathcal{A}$ *its associated infinitesimal generator given by (4). For all* $1 \leq n \leq N_0$*, we define*

- $\mathcal{B}_n = (e_{n,l})_{1\leq l\leq d_n}$ *an orthonormal basis of* $R(\Pi_n)$*;*
- $\overline{A}_n \in \mathbb{R}^{d_n \times d_n}$ *the matrix of* $A_n|_{R(\Pi_n)} : R(\Pi_n) \to R(\Pi_n)$ *in the orthonormal basis* $\mathcal{B}_n$*;*
- $\overline{A} = \mathrm{diag}(\overline{A}_1, \ldots, \overline{A}_{N_0}) \in \mathbb{R}^{d\times d}$ *where* $d = \sum_{n=1}^{N_0} d_n$.

We constrain the structure of the control law as follows:

$$u(t) = \sum_{n=1}^{N_0} \sum_{l=1}^{d_n} u_{n,l}(t) e_{n,l} \in \bigoplus_{n=1}^{N_0} R(\Pi_n). \tag{9}$$

Introducing

$$\begin{aligned} \overline{x}(t) &= \begin{bmatrix} \overline{x}_1(t)^\top & \overline{x}_2(t)^\top & \ldots & \overline{x}_{N_0}(t)^\top \end{bmatrix}^\top \in \mathbb{R}^d, \\ \overline{u}(t) &= \begin{bmatrix} \overline{u}_1(t)^\top & \overline{u}_2(t)^\top & \ldots & \overline{u}_{N_0}(t)^\top \end{bmatrix}^\top \in \mathbb{R}^d \end{aligned}$$

[1] In [11, Lem. 2.1], this assumption is replaced by $\|e^{A_n t}\| \leq g(t)$ for all $t \geq 0$ and $n \geq 1$. However, as $\Pi_n^2 = \Pi_n$, the proof reported in [11, Lem. 2.1] also applies in a direct manner to the setting used in this paper.

[2] This implies that we have a finite number of unstable modes (counted with multiplicity). The same remark applies to the assumptions of Theorem 2.

with

$$\overline{x}_n(t) = \begin{bmatrix} x_{n,1}(t) & x_{n,2}(t) & \dots & x_{n,d_n}(t) \end{bmatrix}^\top \in \mathbb{R}^{d_n},$$
$$\overline{u}_n(t) = \begin{bmatrix} u_{n,1}(t) & u_{n,2}(t) & \dots & u_{n,d_n}(t) \end{bmatrix}^\top \in \mathbb{R}^{d_n}$$

and $x_n(t) = \sum_{l=1}^{d_n} x_{n,l}(t) e_{n,l}$*, the control input takes the form*

$$\overline{u}(t) = \varphi(t) K \left\{ e^{D_0 \overline{A}} \overline{x}(t) + \int_{t-D_0}^{t} e^{(t-s)\overline{A}} \overline{u}(s) \, \mathrm{d}s \right\} \tag{10}$$

where $\varphi \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$ *is an arbitrary transition signal*[3] *over* $[0,t_0]$ *and* $K \in \mathbb{R}^{d \times d}$ *is a feedback gain such that* $A_{\mathrm{cl}} = \overline{A} + K$ *is Hurwitz. Considering* $\Theta_{\underline{u}}(\delta,\kappa)$ *given by (15) in the Appendix with* $M = A_{\mathrm{cl}}$ *and* $N = e^{D_0 \overline{A}} K$*, let* $\delta \in (0, D_0)$ *and* $\kappa \in (0,\sigma]$ *be such that*[4] *the LMI* $\Theta_u(\delta,\kappa) \prec 0$ *is feasible for some* $P_1, Q \in \mathbb{S}_d^{+*}$ *and* $P_2, P_3 \in \mathbb{R}^{d\times d}$*. Then there exists* $C_0 > 0$ *such that, for any* $X_0 \in \mathcal{H}$ *and any* $D \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+)$ *with* $|D - D_0| \leq \delta$*, the mild solutions of (8) with command input* u *given by (9) satisfy*

$$\|X(t)\| + \|u(t)\| \leq C_0 e^{-\kappa t} \|X_0\|$$

for all $t \geq 0$*.*

Proof. First, since A_n is bounded, we have $\|e^{A_n t}\Pi_n\| \leq e^{\|A_n\| t}$ for all $1 \leq n \leq N_0$ and $t \geq 0$. Since we assumed that $\|e^{A_n t}\Pi_n\| \leq M e^{-\sigma t}$ for all $n \geq N_0 + 1$ and $t \geq 0$, we have the existence of a continuous function g such that (2) holds. Thus the conclusions of Lemma 1 apply.

As $D(t) \geq D_0 - \delta > 0$, the well-posedness of the closed-loop system with $u \in \mathcal{C}^0(\mathbb{R}_+;\mathcal{H})$ is a straightforward consequence of the application of the steps method and the fact that the solution of the fixed-point equation (10) is uniquely defined with $\overline{u} \in \mathcal{C}^0(\mathbb{R};\mathbb{R}^d)$ and $u(\tau) = 0$ for $\tau \leq 0$; see [1] for details.

From (7) and (9), we have for all $t \geq 0$ and all $1 \leq n \leq N_0$ that

$$\overline{x}_n(t) = e^{\overline{A}_n t} \overline{x}_n(0) + \int_0^t e^{\overline{A}_n (t-s)} \overline{u}_n(s - D(s)) \mathrm{d}s.$$

As $\overline{u}_n$ and D are continuous, we infer that $\overline{x}_n$ is continuously differentiable and satisfies the ordinary differential equation (ODE)

$$\dot{\overline{x}}_n(t) = \overline{A}_n \overline{x}_n(t) + \overline{u}_n(t - D(t))$$

for all $t \geq 0$. Thus we have

$$\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{u}(t - D(t))$$

for all $t \geq 0$. The application of Theorem 3 reported in Appendix shows that $\|\overline{x}(t)\| + \|\overline{u}(t)\| \leq C_1 e^{-\kappa t} \|\overline{x}(0)\|$ for all $t \geq 0$. Noting that

$$\|\overline{x}(t)\|^2 = \sum_{n=1}^{N_0} \|\overline{x}_n(t)\|^2 = \sum_{n=1}^{N_0} \sum_{l=1}^{d_n} |x_{n,l}(t)|^2 = \sum_{n=1}^{N_0} \|x_n(t)\|^2,$$

we infer that

$$\sum_{n=1}^{N_0} \|x_n(t)\|^2 \leq C_1^2 e^{-2\kappa t} \sum_{n=1}^{N_0} \|x_n(0)\|^2$$

and

$$\|u(t)\| = \sqrt{\sum_{n=1}^{N_0} \sum_{l=1}^{d_n} |u_{n,l}(t)|^2} = \|\overline{u}(t)\|$$
$$\leq C_1 e^{-\kappa t} \sqrt{\sum_{n=1}^{N_0} \|x_n(0)\|^2} \leq C_1 e^{-\kappa t} \|X_0\|.$$

Now, from (9) we see that $\Pi_n u(t) = 0$ for all $n \geq N_0 + 1$ and all $t \geq 0$. Then we have from (7) that $x_n(t) = e^{A_n t} x_n(0) = e^{A_n t} \Pi_n x_n(0)$ for all $n \geq N_0 + 1$ and all $t \geq 0$, where the latter identity holds because $x_n(0) \in R(\Pi_n)$ and thus $\Pi_n x_n(0) = x_n(0)$, hence

$$\sum_{n \geq N_0+1} \|x_n(t)\|^2 \leq M^2 e^{-2\sigma t} \sum_{n \geq N_0+1} \|x_n(0)\|^2.$$

Introducing $C_2 = \max(C_1, M) \geq 1$ and recalling that $0 < \kappa \leq \sigma$, we infer from the latter estimates that

$$\|X(t)\|^2 = \sum_{n \geq 1} \|x_n(t)\|^2 \leq C_2^2 e^{-2\kappa t} \sum_{n\geq 1} \|x_n(0)\|^2$$
$$\leq C_2^2 e^{-2\kappa t} \|X(0)\|^2$$

for all $t \geq 0$. This completes the proof. □

B. Example of application

We consider in this section the case of the following reaction-diffusion equation with Robin boundary conditions:

$$y_t(t,x) = a y_{xx}(t,x) + b y(t,x) + u(t - D(t), x) \tag{11a}$$
$$\cos(\theta_1) y(t,0) - \sin(\theta_1) y_x(t,0) = 0 \tag{11b}$$
$$\cos(\theta_2) y(t,1) + \sin(\theta_2) y_x(t,1) = 0 \tag{11c}$$
$$y(0,x) = \phi(x), \tag{11d}$$

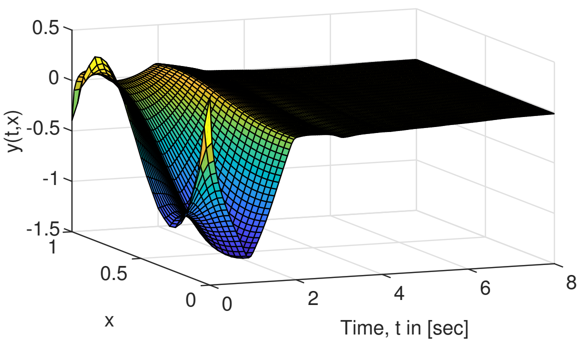
for $t > 0$ and $x \in (0,1)$. Such a problem was considered in [23] in the case of Dirichlet boundary conditions and for a controller designed via a backstepping transformation. Here we have $a > 0$, $b \in \mathbb{R}$, and $\theta_1, \theta_2 \in [0, 2\pi)$. In this setting, $u : [-D_0 - \delta, +\infty) \times (0,1) \to \mathbb{R}$ with $u(t,\cdot) = 0$ for $t < 0$ is the control input subject to an uncertain and time-varying input delay $D \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$ with $|D - D_0| \leq \delta$ and where $D_0 > 0$ and $\delta \in (0, D_0)$ are known constants. Finally, $\phi \in L^2(0,1)$ represents the initial condition. This type of setting (in-domain control of reaction-diffusion processes) occurs, e.g., in the context of nuclear fusion with the confinement of a hot plasma in a Tokamak [17]

The reaction-diffusion system (11) can be rewritten under the form (8) over the state-space $\mathcal{H} = L^2(0,1)$ endowed with its usual inner product $\langle f, g \rangle = \int_0^1 f(\xi) g(\xi) \, \mathrm{d}\xi$. In this setting, $X(t) = y(t,\cdot) \in \mathcal{H}$, $X_0 = \phi \in \mathcal{H}$, and $\mathcal{A}f = af'' + bf \in \mathcal{H}$ defined on the domain $D(\mathcal{A}) = \{ f \in H^2(0,1) \,:\, \cos(\theta_1) f(0) - \sin(\theta_1) f'(0) = 0, \, \cos(\theta_2) f(1) + \sin(\theta_2) f'(1) = 0 \}$. From the Sturm-Liouville theory (see, e.g., [24, Sec. 8.6]), it is well known that $\mathcal{A}$ is a self-adjoint operator with compact resolvent whose eigenvalues $(\lambda_n)_{n\geq 1}$ are all real and can be sorted to form a strictly decreasing sequence with $\lambda_n \to -\infty$ when $n \to +\infty$ and associated unit eigenvectors e_n can be selected such that $(e_n)_{n\geq 1}$ forms a Hilbert basis of $\mathcal{H}$. In order to carry out the computations, we assume that $h_i = \cot(\theta_i) > 0$. This corresponds to the most common Robin boundary conditions encountered in practical applications (e.g., convection boundary conditions for heat equations). Standard computations show that $\lambda_n = b - a r_n^2$ for $n \geq 1$, where $(r_n)_{n\geq 1}$ is the increasing sequence formed by the (strictly) positive solutions r of $\left(h_1 h_2 - r^2\right)\sin(r) + (h_1 + h_2) r \cos(r) = 0$. The corresponding unit eigenvectors are given by $e_n = \varphi_n / \|\varphi_n\|$ with $\varphi_n(x) = r_n \cos(r_n x) + h_1 \sin(r_n x)$.

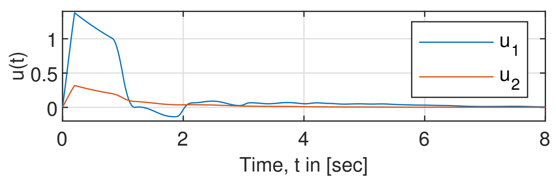
Then, one can show that $\mathcal{A}$ is a Riesz-spectral operator [2, Def. 2.3.4] and generates a C_0-semigroup denoted by $T(t)$ [3]. The result reported in [2, Thm. 2.3.5] shows that $T(t)$ and $\mathcal{A}$ can be written as (3) and (4), respectively with operators $A_n, \Pi_n \in \mathcal{L}(\mathcal{H})$ given by $A_n z = \lambda_n \langle z, e_n \rangle e_n$ and $\Pi_n z = \langle z, e_n \rangle e_n$ for all $z \in \mathcal{H}$. Moreover, it is readily checked that $(A_n)_{n\geq 1}$ is compatible with the family of complete orthogonal projections $(\Pi_n)_{n\geq 1}$. Furthemore,

[3] See notations at the beginning of Section II

[4] The considered LMI is always feasible for sufficiently small values of $\delta > 0$ and $\kappa > 0$. See Appendix and [13, Lem. 2] for details.



(a) State $y(t, x)$



(b) Components u_1, u_2 of the distributed command input u

Fig. 1. Time evolution of the closed-loop reaction-diffusion system

noting that $e^{A_n t}\Pi_n z = e^{\lambda_n t}\langle z, e_n\rangle e_n$, we obtain that $\|e^{A_n t}\Pi_n\| \leq e^{\lambda_1 t}$ for all $t \geq 0$. Then the conclusions of Lemma 1 apply.

We select N_0 as the largest integer such that $\lambda_{N_0} \geq 0$. Then we have that $d_n = \dim(R(\Pi_n)) = 1 < +\infty$ for all $n \geq 1$ and that, as $e^{A_n t}\Pi_n z = e^{\lambda_n t}\langle z, e_n\rangle e_n$, $\|e^{A_n t}\Pi_n\| \leq e^{\lambda_{N_0+1} t}$ for all $n \geq N_0 + 1$ with $\lambda_{N_0+1} < 0$. Hence the assumptions of Theorem 1 are satisfied for values of $\delta \in (0, D_0)$ and $\kappa \in (0, \sigma)$ such that the LMI $\Theta_u(\delta, \kappa) \prec 0$ with $M = A_{\mathrm{cl}}$ and $N = e^{D_0 \overline{A}} K$ is feasible, assessing the exponential stability of the resulting closed-loop system.

For numerical computations, we set $a = 0.05$, $b = 1$, $\theta_1 = \pi/3$, and $\theta_2 = \pi/10$. The nominal value of the delay is set as $D_0 = 1\,\mathrm{s}$. The first three eigenvalues of $\mathcal{A}$ are approximately given by $\lambda_1 \approx 0.8890$, $\lambda_2 \approx 0.2203$, and $\lambda_3 \approx -1.3075$. Thus we take $N_0 = 2$ and we select the feedback gain as $K = \mathrm{diag}(-0.5, -0.75) - \overline{A}$ such that the closed-loop matrix $A_{\mathrm{cl}} = \overline{A} + K = \mathrm{diag}(-0.5, -0.75)$ is Hurwitz. The control input takes the form $u(t, x) = u_1(t)e_1(x) + u_2(t)e_2(x)$ where $\overline{u}(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^\top$ is given by (10). The application of Theorem 1 shows that the closed-loop system is exponentially stable for $\delta = 0.209$. We note that the structure of the obtained controller in significantly simpler than the one reported in [23] in the special case of Dirichlet boundary conditions. The reason is that, in the present work, only a finite number of modes of the original reaction-diffusion equation are actively controlled.

The numerical behavior of the closed-loop system (obtained based on the 30 dominant modes) with the initial condition $\phi(x) = (1 - 2x)/2 + 20x(1-x)(x-3/5)$, the time-varying delay $D(t) = 1 + 0.2\sin(5t)$, and the transition signal φ over $[0, t_0]$ with $t_0 = 0.2\,\mathrm{s}$ and linearly increasing from 0 to 1 on $[0, t_0]$, is shown in Fig. 1. The numerical results are compliant with the theoretical predictions.

IV. Finite dimensional control input acting via a bounded operator

We now consider the case of a finite-dimensional control input acting over the domain via a bounded operator. Specifically, we consider the abstract system (6) with $f(t) = \sum_{k=1}^m f_k u_k(t - D_k(t))$ where $f_k \in \mathcal{H}$, $u_k(t) \in \mathbb{R}$ is a scalar control input subject to an uncertain and time-varying input-delay $D_k(t) \in [D_{0,k} - \delta_k, D_{0,k} + \delta_k]$. We assume the system uncontrolled in negative times, i.e. $u(\tau) = 0$ for $\tau \leq 0$. Thus, the studied system takes the following form:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + \sum_{k=1}^m f_k u_k(t - D_k(t)) \tag{12a}$$

$$X(0) = X_0 \tag{12b}$$

The above setting differs from (8) because 1) the structure of the distributed input (number of scalar inputs m and the functions f_k) is imposed *a priori* and hence cannot be tuned at will during control design; 2) each scalar input u_k exhibits a distinct input delay D_k. This latter element yields a difference of treatment in the stability analysis compared to the case of a uniform delay, inducing different LMI conditions (see Appendix).

A. Main result

The main result of this section is the following theorem.

Theorem 2: *Let $D_{0,k}, t_0 > 0$ be given. Let $(A_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$ be compatible with $(\Pi_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$ a family of complete orthogonal projections. We assume that there exists $N_0 \geq 1$ such that:*

- *$d_n = \dim(R(\Pi_n)) < \infty$ for all $1 \leq n \leq N_0$;*
- *there exist $M \geq 1$ and $\sigma > 0$ such that $\|e^{A_n t}\Pi_n\| \leq Me^{-\sigma t}$ for all $n \geq N_0 + 1$ and $t \geq 0$.*

We denote by $T(t)$ the C_0-semigroup defined by (3) and we consider $\mathcal{A}$ its associated infinitesimal generator given by (4). For all $1 \leq n \leq N_0$, we define

- *$\mathcal{B}_n = (e_{n,l})_{1\leq l\leq d_n}$ an orthonormal basis of $R(\Pi_n)$;*
- *$\overline{A}_n \in \mathbb{R}^{d_n \times d_n}$ the matrix of $A_n|_{R(\Pi_n)} : R(\Pi_n) \to R(\Pi_n)$ in the orthonormal basis $\mathcal{B}_n$;*
- *$\overline{A} = \mathrm{diag}(\overline{A}_1, \ldots, \overline{A}_{N_0}) \in \mathbb{R}^{d\times d}$ where $d = \sum_{n=1}^{N_0} d_n$;*
- *$\overline{B}_n = (\langle f_k, e_{n,l}\rangle)_{1\leq l\leq d_n,\, 1\leq k\leq m} \in \mathbb{R}^{d_n\times m}$;*
- *$\overline{B} \in \mathbb{R}^{d\times m} = \begin{bmatrix} \overline{B}_1^\top & \ldots & \overline{B}_{N_0}^\top \end{bmatrix}^\top \in \mathbb{R}^{d\times m}$ whose k-th column is denoted by $\tilde{B}_k \in \mathbb{R}^d$.*

We assume that the pair $(\overline{A}, \overline{B})$ is stabilizable. Introducing

$$\overline{x}(t) = \begin{bmatrix} \overline{x}_1(t)^\top & \overline{x}_2(t)^\top & \ldots & \overline{x}_{N_0}(t)^\top \end{bmatrix}^\top \in \mathbb{R}^d$$

with

$$\overline{x}_n(t) = \begin{bmatrix} x_{n,1}(t) & x_{n,2}(t) & \ldots & x_{n,d_n}(t) \end{bmatrix}^\top \in \mathbb{R}^{d_n}$$

and $x_n(t) = \sum_{l=1}^{d_n} x_{n,l}(t)e_{n,l}$, the control input takes the form

$$u(t) = \varphi(t)K\left\{\overline{x}(t) + \sum_{k=1}^m \int_{t-D_{0,k}}^t e^{(t-D_{0,k}-s)\overline{A}}\tilde{B}_k u(s)\,\mathrm{d}s\right\} \tag{13}$$

where $\varphi \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R})$ is an arbitrary transition signal over $[0, t_0]$ and $K \in \mathbb{R}^{m\times d}$ is selected such that $A_{\mathrm{cl}} \triangleq \overline{A} + \sum_{k=1}^m e^{-D_{0,k}\overline{A}}\tilde{B}_k K_k$ is Hurwitz with $K_k \in \mathbb{R}^{1\times d}$ the k-th line of K. Considering $\Theta_d(\Delta, \kappa)$ given by (16) in the Appendix with $\Delta = (\delta_1, \ldots, \delta_m)$, $M = A_{\mathrm{cl}}$, and $N_k = \tilde{B}_k K_k$, let $\delta_k \in (0, D_{0,k})$ and $\kappa \in (0, \sigma)$ be such that[5] *the LMI $\Theta_d(\Delta, \kappa) \prec 0$ is feasible for some $P_1, Q_k \in \mathbb{S}_d^{+*}$ and $P_2, P_3 \in \mathbb{R}^{d\times d}$. Then there exists $C_0 > 0$ such that, for any $X_0 \in \mathcal{H}$ and any $D_k \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R}_+)$ with $|D_k - D_{0,k}| \leq \delta_k$, the mild solutions of (12) with command input u given by (13) satisfy*

$$\|X(t)\| + \|u(t)\| \leq C_0 e^{-\kappa t}\|X_0\|$$

for all $t \geq 0$.

Proof. The same argument as the one used in the proof of Theorem 1 shows that 1) the conclusions of Lemma 1 apply; and

[5] We recall that the considered LMI is always feasible for sufficiently small values of $\delta_k > 0$ and $\kappa > 0$. See Appendix and [14] for details.

2) since $D_k(t) \geq D_{0,k} - \delta_k > 0$, the closed-loop system is well-posed and we have $u \in \mathcal{C}^0(\mathbb{R};\mathbb{R}^m)$ with $u(\tau) = 0$ for $\tau \leq 0$.

Setting $f(t) = \sum_{k=1}^m f_k u_k(t - D_k(t))$ with $f_k \in \mathcal{H}$ and $u_k(t) \in \mathbb{R}$, we evaluate for $1 \leq n \leq N_0$ the following term:

$$\begin{aligned}
&\int_0^t e^{A_n(t-s)} \Pi_n f(s) \,\mathrm{d}s \\
&= \int_0^t e^{A_n(t-s)} \Pi_n \sum_{k=1}^m f_k u_k(s - D_k(s)) \,\mathrm{d}s \\
&= \int_0^t e^{A_n(t-s)} \sum_{k=1}^m \left(\sum_{l=1}^{d_n} \langle f_k, e_{n,l} \rangle \, e_{n,l} \right) u_k(s - D_k(s)) \,\mathrm{d}s \\
&= \int_0^t e^{A_n(t-s)} \sum_{l=1}^{d_n} \sum_{k=1}^m \left\{ \langle f_k, e_{n,l} \rangle \, u_k(s - D_k(s)) \right\} e_{n,l} \,\mathrm{d}s \\
&= \int_0^t e^{A_n(t-s)} \sum_{l=1}^{d_n} \overline{B}_{n,l} v(s) e_{n,l} \,\mathrm{d}s
\end{aligned}$$

where $v(t) = \begin{bmatrix} u_1(t - D_1(t)) & \dots & u_m(t - D_m(t)) \end{bmatrix}^\top \in \mathbb{R}^m$ and $\overline{B}_{n,l}$ is the l-th line of $\overline{B}_n$. From (7), we have for all $t \geq 0$ and all $1 \leq n \leq N_0$ that

$$x_n(t) = e^{A_n t} x_n(0) + \int_0^t e^{A_n(t-s)} \sum_{l=1}^{d_n} \overline{B}_{n,l} v(s) e_{n,l} \,\mathrm{d}s$$

and thus

$$\overline{x}_n(t) = e^{\overline{A}_n t} \overline{x}_n(0) + \int_0^t e^{\overline{A}_n(t-s)} \overline{B}_n v(s) \mathrm{d}s.$$

As v is continuous, we infer that $\overline{x}_n$ is continuously differentiable and satisfies the ODE

$$\dot{\overline{x}}_n(t) = \overline{A}_n \overline{x}_n(t) + \overline{B}_n v(t)$$

for all $t \geq 0$. Then we have

$$\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}v(t) = \overline{A}\overline{x}(t) + \sum_{k=1}^m \tilde{B}_k u_k(t - D_k(t))$$

for all $t \geq 0$. The application of Theorem 4 reported in Appendix shows that $\|\overline{x}(t)\| + \|u(t)\| \leq C_1 e^{-\kappa t} \|\overline{x}(0)\|$ for all $t \geq 0$. Noting that

$$\|\overline{x}(t)\|^2 = \sum_{n=1}^{N_0} \|\overline{x}_n(t)\|^2 = \sum_{n=1}^{N_0} \sum_{l=1}^{d_n} |x_{n,l}(t)|^2 = \sum_{n=1}^{N_0} \|x_n(t)\|^2,$$

we infer that

$$\sum_{n=1}^{N_0} \|x_n(t)\|^2 \leq C_1^2 e^{-2\kappa t} \sum_{n=1}^{N_0} \|x_n(0)\|^2$$

and

$$\|u(t)\| \leq C_1 e^{-\kappa t} \sqrt{\sum_{n=1}^{N_0} \|x_n(0)\|^2} \leq C_1 e^{-\kappa t} \|X_0\|.$$

Now, from (7) and recalling that $\Pi_n^2 = \Pi_n$, we have for all $n \geq N_0 + 1$ and all $t \geq 0$ that

$$\begin{aligned}
\|x_n(t)\| &\leq M e^{-\sigma t} \|x_n(0)\| \\
&\quad + M \int_0^t e^{-\sigma(t-s)} \sum_{k=1}^m \|\Pi_n f_k\| |u_k(s - D_k(s))| \,\mathrm{d}s
\end{aligned}$$

As $u(t) = 0$ for $t \leq 0$, we note that

$$\begin{aligned}
|u_k(t - D_k(t))| \leq \|u(t - D_k(t))\| &\leq C_1 e^{-\kappa(t - D_k(t))} \|X_0\| \\
&\leq C_2 e^{-\kappa t} \|X_0\|
\end{aligned}$$

for all $t \geq 0$ with $C_2 = C_1 \exp\left(\kappa \max_{1 \leq k \leq m} \{ D_{0,k} + \delta_k \} \right)$. Then we obtain that

$$\begin{aligned}
&\|x_n(t)\| \\
&\leq M e^{-\sigma t} \|x_n(0)\| + M C_2 \sum_{k=1}^m \|\Pi_n f_k\| e^{-\sigma t} \int_0^t e^{(\sigma - \kappa)s} \,\mathrm{d}s \, \|X_0\| \\
&\leq M e^{-\sigma t} \|x_n(0)\| + \frac{M C_2}{\sigma - \kappa} \sum_{k=1}^m \|\Pi_n f_k\| e^{-\kappa t} \|X_0\|
\end{aligned}$$

where we used $0 < \kappa < \sigma$. Consequently we have

$$\begin{aligned}
&\sum_{n \geq N_0 + 1} \|x_n(t)\|^2 \\
&\leq 2 M^2 e^{-2\sigma t} \sum_{n \geq N_0 + 1} \|x_n(0)\|^2 \\
&\quad + \frac{2 m M^2 C_2^2}{(\sigma - \kappa)^2} \sum_{k=1}^m \sum_{n \geq N_0 + 1} \|\Pi_n f_k\|^2 e^{-2\kappa t} \|X_0\|^2 \\
&\leq 2 M^2 e^{-2\kappa t} \sum_{n \geq N_0 + 1} \|x_n(0)\|^2 \\
&\quad + \frac{2 m M^2 C_2^2}{(\sigma - \kappa)^2} \sum_{k=1}^m \|f_k\|^2 e^{-2\kappa t} \|X_0\|^2.
\end{aligned}$$

Introducing the constant $C_3 \geq 1$ defined by

$$C_3^2 = \max\left(C_1^2, 2M^2 \right) + \frac{2 m M^2 C_2^2}{(\sigma - \kappa)^2} \sum_{k=1}^m \|f_k\|^2,$$

we obtain that, for all $t \geq 0$,

$$\|X(t)\|^2 = \sum_{n \geq 1} \|x_n(t)\|^2 \leq C_3^2 e^{-2\kappa t} \|X(0)\|^2.$$

This completes the proof. □

B. Example of application

We consider a clamped flexible string described by

$$\begin{aligned}
y_{tt}(t,x) &= (\alpha y_x + \beta y_{tx})_x(t,x) + \gamma y(t,x) \\
&\quad + \sum_{k=1}^m \zeta_k(x) u_k(t - D_k(t)) && \text{(14a)} \\
y(t,0) &= y(t,1) = 0 && \text{(14b)} \\
y(0,x) &= \phi_1(x), \quad y_t(0,x) = \phi_2(x), && \text{(14c)}
\end{aligned}$$

for $t > 0$ and $x \in (0,1)$. Here we have $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$, and $\zeta_k \in L^2(0,1)$. In this setting, $u_k : [-D_{0,k} - \delta_k, +\infty) \to \mathbb{R}$ with $u_k(t) = 0$ for $t < 0$ is the control input subject to an uncertain and time-varying input delay $D_k \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$ with $|D_k - D_{0,k}| \leq \delta_k$ and where $D_{0,k} > 0$ and $\delta_k \in (0, D_{0,k})$ are known constants. Finally, $\phi_1 \in H_0^1(0,1)$ and $\phi_2 \in L^2(0,1)$ represent the initial conditions.

The flexible dynamics described by (14) can be rewritten under the form (12) over the state-space $\mathcal{H} = H_0^1(0,1) \times L^2(0,1)$ endowed with the inner product $\langle (g_1,h_1),(g_2,h_2) \rangle = \int_0^1 \alpha g_1'(\xi) \overline{g_2'(\xi)} + h_1(\xi) \overline{h_2(\xi)} \,\mathrm{d}\xi$. In this setting, $X(t) = (y(t,\cdot), y_t(t,\cdot)) \in \mathcal{H}$, $X_0 = (\phi_1,\phi_2) \in \mathcal{H}$, $f_k = (0,\zeta_k) \in \mathcal{H}$, and $\mathcal{A}(g,h) = (h, (\alpha g' + \beta h')' + \gamma g) \in \mathcal{H}$ defined on the domain:

$$D(\mathcal{A}) = \left\{ (g,h) \in \mathcal{H} \,:\, h \in H_0^1(0,1),\ \alpha g' + \beta h' \in H^1(0,1) \right\}.$$

Introducing $\mathcal{A}_0(g,h) = (h, (\alpha g' + \beta h')') \in \mathcal{H}$ defined on the domain $D(\mathcal{A}_0) = D(\mathcal{A})$ and $L \in \mathcal{L}(\mathcal{H})$ defined by[6] $L(g,h) = (0, \gamma g)$, we have $\mathcal{A} = \mathcal{A}_0 + L$. As $\beta > 0$, an integration by parts shows

[6]The bounded nature follows from Poincaré's inequality.

that $\mathcal{A}_0$ is dissipative while standard computations yield $0 \in \rho(\mathcal{A}_0)$. Thus, by the Lümer-Phillips theorem, $\mathcal{A}_0$ generates a C_0-semigroup of contractions. Then $\mathcal{A}$ generates a C_0-semigroup denoted by $T(t)$.

For $n \geq 1$ we introduce the vectors

$$e_{n,1} = \frac{1}{n\pi}\sqrt{\frac{2}{\alpha}}\,(\sin(n\pi\cdot), 0) \in D(\mathcal{A}),$$
$$e_{n,2} = \sqrt{2}\,(0, \sin(n\pi\cdot)) \in D(\mathcal{A}).$$

It is readily checked that $e_{n,1}, e_{n,2}$ are unit vectors that satisfy $\langle e_{n_1,l_1}, e_{n_2,l_2}\rangle = \delta_{(n_1,l_1),(n_2,l_2)} \in \{0,1\}$ with $\delta_{(n_1,l_1),(n_2,l_2)} = 1$ if and only if $(n_1,l_1) = (n_2,l_2)$. Then $(e_{n,l})_{n\geq 1, 1\leq l\leq 2}$ is an orthonormal family of vectors of $\mathcal{H}$. Let us check that this family is also complete. Let $(g,h) \in \mathcal{H}$ be such that $\langle (g,h), e_{n,l}\rangle = 0$ for all $n \geq 1$ and $1 \leq l \leq 2$. We deduce that $\langle g', \cos(n\pi\cdot)\rangle_{L^2(0,1)} = 0$ and $\langle h, \sin(n\pi\cdot)\rangle_{L^2(0,1)} = 0$ for all $n \geq 1$. Recalling that $\{1, \sqrt{2}\cos(n\pi\cdot),\ n \geq 1\}$ and $\{\sqrt{2}\sin(n\pi\cdot),\ n \geq 1\}$ are Hilbert basis of $L^2(0,1)$, we infer that g' is a constant function while $h = 0$. As $g \in H_0^1(0,1)$, we have that $g = 0$. We conclude that $(e_{n,l})_{n\geq 1, 1\leq l\leq 2}$ is complete and thus forms a Hilbert basis of $\mathcal{H}$.

Now, we note that

$$\mathcal{A}e_{n,1} = \frac{\gamma - \alpha n^2\pi^2}{\sqrt{\alpha}n\pi} e_{n,2},$$
$$\mathcal{A}e_{n,2} = \sqrt{\alpha}n\pi e_{n,1} - \beta n^2\pi^2 e_{n,2}.$$

Then, we can introduce $(\Pi_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$ the family of complete orthogonal projections defined by $\Pi_n z = \langle z, e_{n,1}\rangle\, e_{n,1} + \langle z, e_{n,2}\rangle\, e_{n,2}$ and define $A_n = \Pi_n \mathcal{A} \Pi_n$. It is readily checked that $(A_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})^{\mathbb{N}}$ is compatible with the family of complete orthogonal projections $(\Pi_n)_{n\geq 1}$. Furthermore, noting that $R(\Pi_n) = \mathrm{vect}(e_{n,1}, e_{n,2})$ is a closed subspace included in $D(\mathcal{A})$ that is $\mathcal{A}$-invariant, then [2, Lem. 2.5.4] ensures that $R(\Pi_n)$ is $T(t)$-invariant and $T(t)|_{R(\Pi_n)} = e^{A_n|_{R(\Pi_n)} t} = e^{A_n t}\big|_{R(\Pi_n)}$ for all $n \geq 1$ and $t \geq 0$. We deduce that

$$T(t)z = T(t)\sum_{n\geq 1}\Pi_n z = \sum_{n\geq 1} T(t)\Pi_n z = \sum_{n\geq 1} e^{A_n t}\Pi_n z,$$

which shows that $T(t)$ takes the form of (3). Finally, since $T(t)$ is a C_0-semigroup, there exist $M \geq 1$ and $\omega > 0$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. In particular, $\|e^{A_n t}\Pi_n\| = \|T(t)\Pi_n\| \leq \|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, showing that (2) holds with $g(t) = Me^{\omega t}$. The assumptions of Lemma 1 are satisfied and thus, by uniqueness of the infinitesimal generator, $\mathcal{A}$ can be rewritten under the form (4).

We introduce $\overline{A}_n \in \mathbb{R}^{2\times 2}$ the matrix of $A_n|_{R(\Pi_n)} : R(\Pi_n) \to R(\Pi_n)$ in the orthonormal basis $(e_{n,1}, e_{n,2})$:

$$\overline{A}_n = \begin{bmatrix} 0 & \sqrt{\alpha}n\pi \\ \dfrac{\gamma - \alpha n^2\pi^2}{\sqrt{\alpha}n\pi} & -\beta n^2\pi^2 \end{bmatrix}.$$

The characteristic polynomial of $\overline{A}_n$ is given by $X^2 + \beta n^2\pi^2 X + \alpha n^2\pi^2 - \gamma$, showing that $\overline{A}_n$ is Hurwitz if and only if $\alpha n^2\pi^2 - \gamma > 0$. Thus, introducing the integer $N_1 = \lfloor \frac{1}{\pi}\sqrt{\frac{\gamma}{\alpha}} \rfloor \geq 0$, $\overline{A}_n$ is Hurwitz for all $n \geq N_1 + 1$. We now show that there exist constants $M \geq 1$ and $\sigma > 0$ such that $\|e^{\overline{A}_n t}\| \leq Me^{-\sigma t}$ for all $n \geq N_1 + 1$ and $t \geq 0$. To do so, we define $\eta = \alpha/(2\beta)$ and the symmetric matrix

$$P_n = \begin{bmatrix} \dfrac{\beta}{\sqrt{\alpha}}n\pi & 1 \\ 1 & \dfrac{\beta}{\sqrt{\alpha}}n\pi \end{bmatrix}.$$

The eigenvalues of P_n are given by $\lambda_m(P_n) = \frac{\beta}{\sqrt{\alpha}}n\pi - 1$ and $\lambda_M(P_n) = \frac{\beta}{\sqrt{\alpha}}n\pi + 1$. Then, considering integers $n > \sqrt{\alpha}/(\beta\pi)$, P_n is symmetric positive definite. The computation of

$$Q_n = (q_{n,k,l})_{1\leq k,l\leq 2} = \overline{A}_n^\top P_n + P_n\overline{A}_n + 2\eta P_n$$

yields

$$q_{n,1,1} = -\sqrt{\alpha}n\pi + \frac{2\gamma}{\sqrt{\alpha}n\pi},$$
$$q_{n,2,2} = -\frac{2\beta^2}{\sqrt{\alpha}}n^3\pi^3 + 3\sqrt{\alpha}n\pi,$$
$$q_{n,1,2} = q_{n,2,1} = -\beta n^2\pi^2 + \frac{\beta\gamma}{\alpha} + \frac{\alpha}{\beta}.$$

In particular, one has

$$q_{n,1,1} \sim -\sqrt{\alpha}n\pi < 0,$$
$$q_{n,1,1}q_{n,2,2} - q_{n,1,2}^2 \sim \beta^2 n^4\pi^4 > 0,$$

when $n \to +\infty$. This shows that there exists an integer $N_2 > \sqrt{\alpha}/(\beta\pi)$ large enough such that, for all $n \geq N_2$, P_n is symmetric positive definite and Q_n is symmetric negative definite. In particular, defining $z_n(t) = e^{\overline{A}_n t}z_0$ for an arbitrary $z_0 \in \mathbb{R}^2$, the introduction of $V_n(t) = z_n(t)^\top P_n z_n(t)$ yields $\lambda_m(P_n)\|z_n(t)\|^2 \leq V_n(t) \leq \lambda_M(P_n)\|z_n(t)\|^2$ and $\dot{V}_n(t) \leq -2\eta V_n(t)$ for all $n \geq N_2$ and $t \geq 0$. Then, we infer that

$$\|e^{\overline{A}_n t}\| \leq \sqrt{\frac{\lambda_M(P_n)}{\lambda_m(P_n)}}e^{-\eta t} \leq \sqrt{\frac{\beta N_2\pi + \sqrt{\alpha}}{\beta N_2\pi - \sqrt{\alpha}}}e^{-\eta t}$$

for all $n \geq N_2$ and $t \geq 0$. Recalling that $\overline{A}_n$ is Hurwitz for $n \geq N_1 + 1$, we have for any $N_1 + 1 \leq n \leq N_2 - 1$ the existence of $M_n \geq 1$ and $\eta_n > 0$ such that $\|e^{\overline{A}_n t}\| \leq M_n e^{-\eta_n t}$ for all $t \geq 0$. Setting $M = \max\left(M_{N_1+1}, \ldots, M_{N_2-1}, \sqrt{\frac{\beta N_2\pi + \sqrt{\alpha}}{\beta N_2\pi - \sqrt{\alpha}}}\right) \geq 1$ and $\sigma = \min(\eta_{N_1+1}, \ldots, \eta_{N_2-1}, \eta) > 0$, we obtain that $\|e^{\overline{A}_n t}\| \leq Me^{-\sigma t}$ for all $n \geq N_1 + 1$ and $t \geq 0$.

For control design, we select an arbitrary integer $N_0 \geq N_1$. In order to apply the result of Theorem 2, it remains to assess that the pair $(\overline{A}, \overline{B})$ is stabilizable. We actually show that $(\overline{A}, \overline{B})$ is stabilizable if and only if for any $1 \leq n \leq N_1$ there exists $1 \leq k = k(n) \leq m$ such that $\int_0^1 \zeta_k(\xi)\sin(n\pi\xi)\,\mathrm{d}\xi \neq 0$. We recall that $\zeta_k \in L^2(0,1)$ models the impact of the control input u_k on the system dynamics (14a). Such a result is closely related to the controllability properties studied in [2, Sec. 4.2]. First, we note that $\langle f_k, e_{n,1}\rangle = 0$ and $\langle f_k, e_{n,2}\rangle = \sqrt{2}\int_0^1 \zeta_k(\xi)\sin(n\pi\xi)\,\mathrm{d}\xi$ for all $n \geq 1$ and $1 \leq k \leq m$. We study the existence of possible common eigenvalues λ to $\overline{A}_n$ and $\overline{A}_{n+m}$ for $n, m \geq 1$. In this case, we have $\lambda^2 + \beta n^2\pi^2\lambda + \alpha n^2\pi^2 - \gamma = 0$ and $\lambda^2 + \beta(n+m)^2\pi^2\lambda + \alpha(n+m)^2\pi^2 - \gamma = 0$. The difference of the two latter identities yields $\lambda = -\alpha/\beta$ while the substitution of this result in the former identity gives $\gamma = \alpha^2/\beta^2$. Conversely, is $\gamma = \alpha^2/\beta^2$, then we have $\lambda = -\alpha/\beta$ that satisfies $\lambda^2 + \beta n^2\pi^2\lambda + \alpha n^2\pi^2 - \gamma = 0$ for all $n \geq 1$. Overall, the only possible common eigenvalue λ to $\overline{A}_n$ and $\overline{A}_{n+m}$ for distinct $n, m \geq 1$ occurs in the configuration $\gamma = \alpha^2/\beta^2$ and is given by $\lambda = -\alpha/\beta < 0$, which is stable. Thus, based on the block diagonal structure of $\overline{A}$, the Hautus test shows that $(\overline{A}, \overline{B})$ is not stabilizable if and only if there exists $1 \leq n \leq N_1$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$, and a non-zero $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \mathbb{C}^2$ such that $x^*\overline{A}_n = \lambda x^*$ and $x^*\overline{B}_n = 0$. If we assume that $\int_0^1 \zeta_k(\xi)\sin(n\pi\xi)\,\mathrm{d}\xi = 0$ for all $1 \leq k \leq m$, then such a x exists hence $(\overline{A}, \overline{B})$ is not stabilizable. Conversely, if there exists $1 \leq k \leq m$ such that $\int_0^1 \zeta_k(\xi)\sin(n\pi\xi)\,\mathrm{d}\xi \neq 0$, then the latter implies that $x_2 = 0$ and then the former yields $\sqrt{\alpha}n\pi\overline{x_1} = \lambda\overline{x_2} = 0$, i.e. $x_1 = 0$. We have reached the claimed conclusion: the pair $(\overline{A}, \overline{B})$

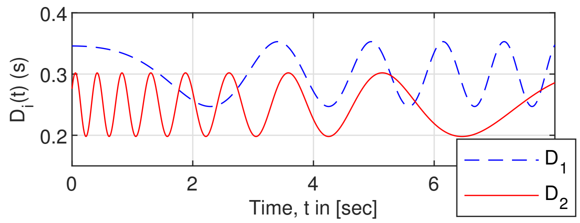
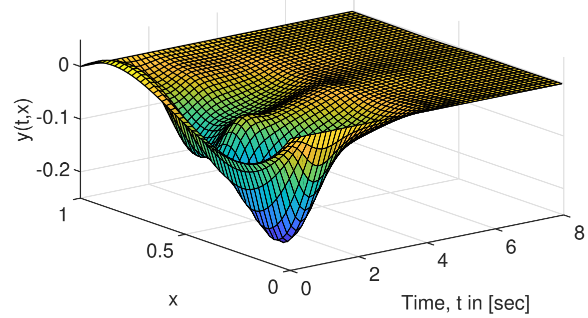


Fig. 2. Time delays $D_k(t)$

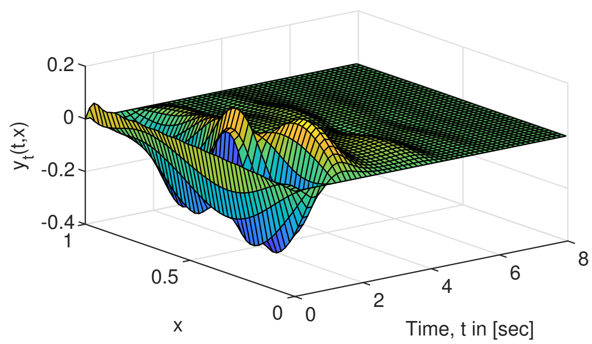
is stabilizable if and only if for any $1 \leq n \leq N_1$ there exists $1 \leq k = k(n) \leq m$ such that $\int_0^1 \zeta_k(\xi)\sin(n\pi\xi)\,\mathrm{d}\xi \neq 0$.

For numerical computations, we set $\alpha = 0.4$, $\beta = 0.05$, and $\gamma = 2\pi^2$. We consider two ($m = 2$) scalar control inputs u_1, u_2 with corresponding distributed actuation $\zeta_1 = 1|_{[1/3,1/2]}$ and $\zeta_2 = 1|_{[5/7,6/7]}$. The nominal values of the delays in the two control input channels are set as $D_{0,1} = 0.3\,\mathrm{s}$ and $D_{0,2} = 0.25\,\mathrm{s}$. In this numerical setting, we have $N_1 = 2$, showing that the matrices $\overline{A}_n$ are Hurwitz for all $n \geq 3$. The eigenvalues of 1) $\overline{A}_1$; 2) $\overline{A}_2$ are approximately given by 1) 3.7347 and -4.2282; 2) 1.2316 and -3.2055. Then we set $N_0 = 2$ for control design. It is readily checked that the pair $(\overline{A}, \overline{B})$ is commandable. Thus, we compute a feedback gain $K \in \mathbb{R}^{2\times 4}$ such that the poles of the closed-loop matrix $A_{\mathrm{cl}} = \overline{A} + \sum_{k=1}^{2} e^{-D_{0,k}\overline{A}}\tilde{B}_k K_k$ are located at -0.8, -1, -3.2, and -4.2. The control input takes the form $u(t,x) = u_1(t)\,1|_{[1/3,1/2]}(x) + u_2(t)\,1|_{[5/7,6/7]}(x)$ where $u(t) = \begin{bmatrix} u_1(t) & u_2(t)\end{bmatrix}^\top$ is given by (13). The application of Theorem 2 shows that the closed-loop system is exponentially stable for $\delta_1 = 0.053$ and $\delta_2 = 0.052$.

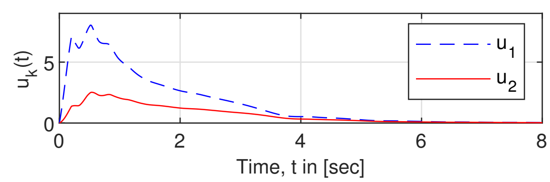
The numerical behavior of the closed-loop system (obtained based on the 40 dominant modes of the system) with the initial condition $\phi_1(x) = -x(2/3 - x)(1 - x)$ and $\phi_2(x) = (1 - \exp(-x))/10$, the time-varying delays as depicted in Fig. 2, and the transition signal φ over $[0, t_0]$ with $t_0 = 0.2\,\mathrm{s}$ and linearly increasing from 0 to 1 on $[0, t_0]$, is shown in Fig. 3. The numerical results are compliant with the theoretical predictions.



(a) State $y(t,x)$



(b) State $y_t(t,x)$



(c) Control input u

Fig. 3. Time evolution of the closed-loop flexible structure

V. Conclusion

This paper discussed the in-domain stabilization of a class of block diagonal infinite-dimensional systems in the presence of uncertain and time-varying control input delays. The proposed control strategy consists in the design of a classical predictor feedback law on a finite-dimensional truncated LTI model capturing the unstable dynamics of the original infinite-dimensional system. Compared to other strategies reported in the literature, this approach offers the advantage that only a finite number of modes of the original infinite-dimensional system is actively controlled, yielding a control strategy with a lower complexity. Moreover, this approach not only holds for fully distributed control inputs, but it also applies to the case of a finite-dimensional control input acting on the domain via a bounded operator. Finally, this approach also allows the consideration of uncertain and time-varying input delays, which are possibly distinct in the different scalar control input channels. The obtained theoretical results were successfully applied to the stabilization of a reaction-diffusion equation with Robin boundary conditions and to a clamped flexible string.

Appendix
Results on the robustness of predictor feedback laws

A. Uniform delay

The following result is extracted from [13].

Theorem 3: *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be such that (A, B) is stabilizable. Let $D_0 > 0$ be a given nominal delay and let φ be an arbitrary transition signal over $[0, t_0]$ with $t_0 > 0$. Let $K \in \mathbb{R}^{m\times n}$ be such that $A_{\mathrm{cl}} \triangleq A + BK$ is Hurwitz. Then, there exists $\delta \in (0, D_0)$ such that for any $D \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+)$ with $|D - D_0| \leq \delta$, the closed-loop system given for $t \geq 0$ by*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - D(t)), \\ u(t) &= \varphi(t) K \left\{ e^{D_0 A} x(t) + \int_{t-D_0}^{t} e^{(t-s)A} B u(s)\,\mathrm{d}s \right\}, \\ x(0) &= x_0, \qquad u(\tau) = 0, \quad -(D_0 + \delta) \leq \tau \leq 0 \end{aligned}$$

with initial condition $x_0 \in \mathbb{R}^n$ is exponentially stable in the sense that there exist constants $\kappa, C_1 > 0$, independent of x_0 and D, such that $\|x(t)\| + \|u(t)\| \leq C_1 e^{-\kappa t}\|x_0\|$. In particular, this conclusion holds true (resp., with given decay rate $\kappa > 0$) for any $\delta \in (0, D_0)$ such that there exist $P_1, Q \in \mathbb{S}_n^{+*}$ and $P_2, P_3 \in \mathbb{R}^{n\times n}$ for which the LMI $\Theta_u(\delta, 0) \prec 0$ (resp., $\Theta_u(\delta, \kappa) \preceq 0$) holds true with

$$\Theta_u(\delta,\kappa) = \qquad (15)$$
$$\begin{bmatrix} 2\kappa P_1 + M^\top P_2 + P_2^\top M & P_1 - P_2^\top + M^\top P_3 & \delta P_2^\top N \\ P_1 - P_2 + P_3^\top M & -P_3 - P_3^\top + 2\delta Q & \delta P_3^\top N \\ \delta N^\top P_2 & \delta N^\top P_3 & -\delta e^{-2\kappa D_0} Q \end{bmatrix}$$

where $M = A_{\mathrm{cl}}$ and $N = e^{D_0 A} BK$.

$$\Theta_d(\Delta,\kappa) = \begin{bmatrix} 2\kappa P_1 + M^\top P_2 + P_2^\top M & P_1 - P_2^\top + M^\top P_3 & \delta_1 P_2^\top N_1 & \delta_2 P_2^\top N_2 & \dots & \delta_m P_2^\top N_m \\ P_1 - P_2 + P_3^\top M & -P_3 - P_3^\top + 2\sum_{k=1}^{m} \delta_k Q_k & \delta_1 P_3^\top N_1 & \delta_2 P_3^\top N_2 & \dots & \delta_m P_3^\top N_m \\ \delta_1 N_1^\top P_2 & \delta_1 N_1^\top P_3 & -\delta_1 e^{-2\kappa D_{0,1}} Q_1 & 0 & \dots & 0 \\ \delta_2 N_2^\top P_2 & \delta_2 N_2^\top P_3 & 0 & -\delta_2 e^{-2\kappa D_{0,2}} Q_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_m N_m^\top P_2 & \delta_m N_m^\top P_3 & 0 & 0 & \dots & -\delta_m e^{-2\kappa D_{0,m}} Q_m \end{bmatrix} \tag{16}$$

Note that due to the Hurwitz nature of A_{cl}, the LMI of Theorem 3 is always feasible for sufficiently small values of $\delta > 0$ and $\kappa > 0$. See [13, Lem. 2] for details.

B. Distinct delays

We have the following result borrowed from [14].

Theorem 4: *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be such that (A,B) is stabilizable. We denote by $B_k \in \mathbb{R}^n$ the k-th column of B. Let $D_{0,k} > 0$ be given nominal delays and let φ be an arbitrary transition signal over $[0,t_0]$ with $t_0 > 0$. Let feedback gains $K_k \in \mathbb{R}^{1\times n}$ be such that $A_{\mathrm{cl}} \triangleq A + \sum_{k=1}^{m} e^{-D_{0,k}A} B_k K_k$ is Hurwitz. We denoted by $K \in \mathbb{R}^{m\times n}$ the feedback gain whose k-th line is K_k. Then, there exist $\delta_k \in (0, D_{0,k})$ such that for any $D_k \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+)$ with $|D_k - D_{0,k}| \leq \delta_k$, the closed-loop system given for $t \geq 0$ by*

$$\begin{aligned} &\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} B_k u_k(t - D_k(t)), \\ &u(t) = \varphi(t) K \left\{ x(t) + \sum_{i=1}^{m} \int_{t-D_{0,i}}^{t} e^{(t-D_{0,i}-s)A} B_i u_i(s) \,\mathrm{s} \right\}, \\ &x(0) = x_0, \qquad u(\tau) = 0, \quad -\max_{1\leq k\leq m} (D_{0,k} + \delta_k) \leq \tau \leq 0 \end{aligned}$$

with initial condition $x_0 \in \mathbb{R}^n$ is exponentially stable in the sense that there exist constants $\kappa, C_1 > 0$, independent of x_0 and D_k, such that $\|x(t)\| + \|u(t)\| \leq C_1 e^{-\kappa t}\|x_0\|$ for all $t \geq 0$. In particular, this conclusion holds true (resp., with given decay rate $\kappa > 0$) for any $\delta_k \in (0, D_{0,k})$ such that there exist $P_1, Q_k \in \mathbb{S}_n^{+*}$ and $P_2, P_3 \in \mathbb{R}^{n\times n}$ for which the LMI $\Theta_d(\Delta, 0) \prec 0$ (resp., $\Theta_d(\Delta,\kappa) \preceq 0$) holds with $\Theta_d(\Delta,\kappa)$ given by (16) where $\Delta = (\delta_1,\dots,\delta_m)$, $M = A_{\mathrm{cl}}$, and $N_k = B_k K_k$.

Similarly to the case of a uniform delay, the LMI of Theorem 4 is always feasible for sufficiently small values of $\delta_k > 0$ and $\kappa > 0$.

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