A Generalized Gaussian Model for Wireless Communications

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***Abstract*—We propose a class of parametric channel models that we call *generalized Gaussian model* (GGM). In particular, given the input, the output is Gaussian with both mean and covariance depending on the input. More general than the conventional linear model, the GGM can capture nonlinearities and self-interference present in more and more wireless communication systems. We focus on three key problems. First, we propose a data-driven model identification algorithm that uses training data to fit the underlying channel with a GGM. This is a generalization of the conventional channel estimation procedure. Second, for an identified GGM, we investigate the receiver design problem and propose several decoding metrics. Third, we are interested in the capacity bounds of the GGM. Both the mismatched lower bound and duality upper bound are proposed. Finally, we apply the GGM to fit the multiple-input multiple-output phase-noise channel. Numerical results show the near optimality of the model identification and decoding algorithms.**

I. Introduction

In wireless communications, a widely accepted channel model is $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x}+\boldsymbol{z}$, where the output $\boldsymbol{y}$ is a linear combination of the input $\boldsymbol{x}$ corrupted by additive Gaussian noise $\boldsymbol{z}$. The linear combination coefficients, represented by the channel matrix $\boldsymbol{H}$, are assumed to be known, or at least can be estimated to reasonable accuracy, at the receiver. Here, the vectors can span over time/frequency/space (antenna) according to the communication system and the encoding scheme. We refer to this model as the linear model or the Gaussian model. Capturing the broadcast and superposition nature of the wireless medium, this model has been adequate in most common scenarios. More importantly, it is analytically tractable, with both known capacity and known capacity achieving communication schemes [1], [2].

Nevertheless, it becomes evident that the Gaussian model does not suffice for emerging communication systems where non-linearities and multiplicative noises are non-negligible. Examples include phase noises due to higher carrier frequencies, hardware impairments in transceivers with large antenna arrays, channel uncertainties due to estimation or quantization noises, etc [3]. Fitting such channels with the conventional linear model is still possible but can be, unsurprisingly, highly suboptimal.

In this paper, we propose to generalize the Gaussian model in such a way that, conditional on the input, the output is still Gaussian, but both the mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{Q}$ of the output can depend on the input. We call this the generalized Gaussian model (GGM). The Gaussian model is thus a special case where the mean is a linear function of the input and the covariance is constant. The GGM can capture both non-linearities and the self-interference caused by multiplicative noises, while remaining Gaussian for tractability. We focus on three key problems.

- *Model identification (channel estimation).* We consider a class of parametric models $(\boldsymbol{\mu}_\theta, \boldsymbol{Q}_\theta)_\theta$. The parameter θ is estimated using training data with the maximum likelihood criterion. The proposed algorithm starts with a least-square initialization and pursues with gradient methods.
- *Receiver design.* For an identified model, we investigate both linear and non-linear detection algorithms. In particular, we consider the linear minimum mean-square estimator (LMMSE), the non-linear zero-forcing (ZF), and an algorithm called self-interference whitening (SIW) [4] based on successive nearest-neighbour decoding.
- *Capacity analysis.* While the exact capacity is hard to obtain even for the simplest GGM, we can derive upper and lower bounds. In particular, we consider the mismatched capacity lower bound [5], [6] for a given input distribution and the proposed decoding metrics. We also propose an upper bound using the duality approach [7].

Finally, we consider an example in which we apply the proposed GGM to fit a multiple-input multiple-output (MIMO) phase noise channel. Numerical results show that the identified model captures accurately the statistical characteristic of this channel. Furthermore, under the identified model, the SIW decoding scheme performs almost optimally in terms of both signal detection error and achievable data rate for QAM inputs. In general, the GGM approximation provides an achievable rate of the underlying channel through the mismatched capacity formulation. The goodness of the approximation can thus be evaluated using the mismatched rate.

Notation: We use lowercase letters with boldface, e.g., $\boldsymbol{v}$, to denote vectors, and uppercase letters with boldface, e.g., $\boldsymbol{M}$, to denote matrices. The Euclidean norm is denoted by $\|\cdot\|$. The trace, transpose, and Moore–Penrose pseudo-inverse of $\boldsymbol{M}$ are respectively $\mathrm{tr}\,(\boldsymbol{M})$, $\boldsymbol{M}^\intercal$, and $\boldsymbol{M}^\dagger$. We use $\mathrm{vec}(\boldsymbol{M})$ to denote the vectorization of $\boldsymbol{M}$ by stacking the columns; $[n] := \{1, 2, \dots, n\}$. The set of $N \times N$ positive definite (semi-definite) matrices is denoted by $\mathbb{S}_{++}^N$ ($\mathbb{S}_{+}^N$). For any $N \times N$ symmetric matrix $\boldsymbol{G}$, we use $(\boldsymbol{G})_+$ to denote the projection to $\mathbb{S}_{+}^N$. (Replacing all negative eigenvalues by 0 provides a L_2 (least-square) projection.) For $\boldsymbol{Q}, \boldsymbol{Q}' \in \mathbb{S}_{+}^N$, $\boldsymbol{Q} \succeq \boldsymbol{Q}'$ means $\boldsymbol{Q} - \boldsymbol{Q}' \in \mathbb{S}_{+}^N$. The Kronecker product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is $\boldsymbol{A} \otimes \boldsymbol{B}$. Logarithms are in base 2.

II. Generalized Gaussian Model

We consider a point-to-point memoryless vector channel such that $p_\theta(\boldsymbol{y}_1,\dots,\boldsymbol{y}_n \,|\, \boldsymbol{x}_1,\dots,\boldsymbol{x}_n) = \prod_{i=1}^n p_\theta(\boldsymbol{y}_i \,|\, \boldsymbol{x}_i)$, where the output at time i, $\boldsymbol{y}_i \in \mathbb{R}^N$, depends only on the input at time i, $\boldsymbol{x}_i \in \mathbb{R}^M$, for $i \in [n]$. The conditional probability density function (pdf) characterizing the channel, $p_\theta(\boldsymbol{y} \,|\, \boldsymbol{x})$, is parameterized by some θ whose value is *a priori* unknown. Here, we assume that $p_\theta(\boldsymbol{y} \,|\, \boldsymbol{x})$ is Gaussian with mean $\boldsymbol{\mu}_\theta(\boldsymbol{x})$ and covariance matrix $\boldsymbol{Q}_\theta(\boldsymbol{x})$, i.e., for $\boldsymbol{y} \in \mathbb{R}^N$

$$p_\theta(\boldsymbol{y} \,|\, \boldsymbol{x}) = \frac{e^{-\frac{1}{2}\mathrm{tr}\left(\boldsymbol{Q}_\theta^{-1}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{\mu}_\theta(\boldsymbol{x}))(\boldsymbol{y}-\boldsymbol{\mu}_\theta(\boldsymbol{x}))^\intercal\right)}}{\sqrt{\det(2\pi \boldsymbol{Q}_\theta(\boldsymbol{x}))}}. \tag{1}$$

Both the mean $\boldsymbol{\mu}_\theta : \mathbb{R}^M \to \mathbb{R}^N$ and the covariance matrix $\boldsymbol{Q}_\theta : \mathbb{R}^M \to \mathbb{S}_{++}^N$ depend on θ. In particular, we assume that $\boldsymbol{\mu}_\theta(\boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{Q}_\theta(0) = \Xi_\theta$ where $\Xi_\theta \in \mathbb{S}_{++}^N$ is the covariance of the measurement noise present irrespective of the input, and $\boldsymbol{Q}_\theta(\boldsymbol{x}) \succeq \boldsymbol{Q}_\theta(\boldsymbol{0})$, $\forall \boldsymbol{x}$. It is thus without loss of generality to assume that $\boldsymbol{Q}_\theta(\boldsymbol{x}) = \Xi_\theta + \boldsymbol{W}_\theta(\boldsymbol{x})\boldsymbol{W}_\theta(\boldsymbol{x})^\intercal$ where $\boldsymbol{W}_\theta(\boldsymbol{x}) : \boldsymbol{x} \to \mathbb{R}^{N\times N'}$ with some $N' \geq N$ and $\boldsymbol{W}_\theta(\boldsymbol{0}) = \boldsymbol{0}$. Equivalently,

$$\boldsymbol{y} = \boldsymbol{\mu}_\theta(\boldsymbol{x}) + \boldsymbol{W}_\theta(\boldsymbol{x})\tilde{\boldsymbol{z}} + \boldsymbol{B}_\theta \boldsymbol{z}, \tag{2}$$

where $\tilde{\boldsymbol{z}}, \boldsymbol{z}$ are independent and identically distributed (i.i.d.) as $\mathcal{N}(0, \mathbf{I}_N)$; $\boldsymbol{B}_\theta \in \mathbb{R}^{N\times N}$ is such that $\boldsymbol{B}_\theta \boldsymbol{B}_\theta^\intercal = \Xi_\theta$. We call this model the *generalized Gaussian model* (GGM).

Remark 1. *Note that for a given $\boldsymbol{Q}_\theta$, the matrices $(\boldsymbol{B}_\theta, \boldsymbol{W}_\theta)$ are underdetermined. A practical advantage of using $(\boldsymbol{B}_\theta, \boldsymbol{W}_\theta)$ instead is that the positive (semi-)definite constraints are gone.*

If we let $\boldsymbol{\mu}_\theta(\boldsymbol{x}) = \boldsymbol{H}\boldsymbol{x}$ and $\boldsymbol{W}_\theta(\boldsymbol{x}) = \boldsymbol{0}$, then the GGM is the conventional Gaussian channel. The non-coherent Rician channel $\boldsymbol{y} = (\hat{\boldsymbol{H}} + \tilde{\boldsymbol{H}})\boldsymbol{x} + \boldsymbol{z}$ where $\tilde{\boldsymbol{H}}$ has i.i.d. $\mathcal{N}(0, \sigma^2)$ entries is also a GGM, if we let $\boldsymbol{\mu}_\theta(\boldsymbol{x}) = \hat{\boldsymbol{H}}\boldsymbol{x}$ and $\boldsymbol{Q}_\theta(\boldsymbol{x}) = (1 + \sigma^2\|\boldsymbol{x}\|^2)\mathbf{I}$. It turns out that the phase noise channel, with moderate phase noise, can also be approximated with a GGM as shown in [4].

We are particularly interested in the following model defined by a class of linear functions.

Definition 1 (Linear self intereference)**.** *When $\boldsymbol{\mu}_\theta(\boldsymbol{x})$ and $\boldsymbol{W}_\theta(\boldsymbol{x})$ are both linear functions of $\boldsymbol{x}$ such that*

$$\boldsymbol{\mu}_\theta(\boldsymbol{x}) = \boldsymbol{A}_\theta \boldsymbol{x} \quad \text{with } \boldsymbol{A}_\theta \in \mathbb{R}^{N\times M}, \tag{3}$$

$$\boldsymbol{W}_\theta(\boldsymbol{x}) = (\mathbf{I}_N \otimes \boldsymbol{x}^\intercal)\boldsymbol{C}_\theta \quad \text{with } \boldsymbol{C}_\theta \in \mathbb{R}^{MN\times N'}, \tag{4}$$

the GGM is called a linear self interference model. This model depends on the parameter θ through the matrices $\boldsymbol{A}_\theta$, $\boldsymbol{B}_\theta$, and $\boldsymbol{C}_\theta$. If the matrices can be arbitrary, we drop the subscript θ and the model parameter becomes the triple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$.

Remark 2. *The covariance matrix of the GGM corresponding to the aforementioned Rician model is a quadratic function of the input. It can be represented by a linear self interference model. In the 2×2 case, we can set $N' = N = 2$, and $\boldsymbol{W}_\theta(\boldsymbol{x}) = \left[\begin{smallmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{smallmatrix}\right]$. In the 3×3 case, $N' = N = 3$ is not enough.*[1] *We can however set $N' = 4$, and $\boldsymbol{W}_\theta(\boldsymbol{x}) = \left[\begin{smallmatrix} x_1 & -x_2 & -x_3 & 0 \\ x_2 & x_1 & 0 & -x_3 \\ x_3 & 0 & x_1 & x_2 \end{smallmatrix}\right]$.*

[1] Indeed, in general, a linear self interference model with $N' = N$ is not enough to represent a GGM with quadratic covariance function.

III. Model Identification

The first problem is how to fit the underlying channel with a GGM, i.e., estimating the channel by identifying the model parameter θ. As in the conventional channel estimation process, the transmitter sends a known training sequence of T input vectors $\{\boldsymbol{x}_i\}_{i=1}^T$ and the receiver estimates the channel using the corresponding output vectors $\{\boldsymbol{y}_i\}_{i=1}^T$. For simplicity,[2] we use the maximum likelihood (ML) estimation, namely,

$$\theta^* = \arg\max_\theta \prod_{i=1}^T p_\theta(\boldsymbol{y}_i \,|\, \boldsymbol{x}_i) \tag{5}$$

$$= \arg\min_\theta L(\theta), \tag{6}$$

where the objective function $L(\theta)$ is defined as

$$L(\theta) := \frac{1}{T}\sum_{i=1}^T f(\boldsymbol{y}_i - \boldsymbol{\mu}_\theta(\boldsymbol{x}_i), \boldsymbol{Q}_\theta(\boldsymbol{x}_i)), \tag{7}$$

$$\text{with } f(\boldsymbol{M}, \boldsymbol{Q}) := \log\det(\boldsymbol{Q}) + \mathrm{tr}\left(\boldsymbol{Q}^{-1}\boldsymbol{M}\boldsymbol{M}^\intercal\right). \tag{8}$$

Let the samples $\{\boldsymbol{x}_i\}$ be drawn from some alphabet $\mathcal{X}$ that is finite ($|\mathcal{X}| \leq T$) for the training phase. We further assume that $\theta = (\theta_1, \theta_2)$ and that $\boldsymbol{\mu}_\theta$ and $\boldsymbol{Q}_\theta$ depend on θ_1 and θ_2, respectively. Then, $L(\theta)$ can be rewritten as

$$L(\theta) = \sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} f\left(\boldsymbol{Y}_{\boldsymbol{x}} - \boldsymbol{\mu}_{\theta_1}(\boldsymbol{x})\mathbf{1}^\intercal, \boldsymbol{Q}_{\theta_2}(\boldsymbol{x})\right), \tag{9}$$

where $\boldsymbol{Y}_{\boldsymbol{x}} := [\boldsymbol{y}_i : \ i \in [T], \boldsymbol{x}_i = \boldsymbol{x}]$ and the sampling ratio.

$$\pi_{\boldsymbol{x}} := \frac{1}{T}|\{i \in [T] : \boldsymbol{x}_i = \boldsymbol{x}\}|. \tag{10}$$

We also define the sample covariance

$$\bar{\boldsymbol{S}}_{\theta_1}(\boldsymbol{x}) := \frac{1}{T\pi_{\boldsymbol{x}}}\left(\boldsymbol{Y}_{\boldsymbol{x}} - \boldsymbol{\mu}_{\theta_1}(\boldsymbol{x})\mathbf{1}^\intercal\right)\left(\boldsymbol{Y}_{\boldsymbol{x}} - \boldsymbol{\mu}_{\theta_1}(\boldsymbol{x})\mathbf{1}^\intercal\right)^\intercal. \tag{11}$$

While the optimization problem is non-convex in general, we propose to find a local optimum in two steps.

1) Step 1: Initialization with least-square: First, we find θ_1 such that $\boldsymbol{\mu}_{\theta_1}(\boldsymbol{x})$ best fits the sample mean

$$\bar{\boldsymbol{y}}_{\boldsymbol{x}} := \frac{1}{T\pi_{\boldsymbol{x}}}\sum_{i\in[T]:\ \boldsymbol{x}_i = \boldsymbol{x}} \boldsymbol{y}_i \tag{12}$$

in the least-square sense, i.e.,

$$\theta_1^{(0)} = \arg\min_{\theta_1}\sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}}\|\bar{\boldsymbol{y}}_{\boldsymbol{x}} - \boldsymbol{\mu}_{\theta_1}(\boldsymbol{x})\|^2. \tag{13}$$

Then, we find θ_2 such that $\boldsymbol{Q}_{\theta_2}(\boldsymbol{x})$ best fits the sample covariance matrix $\bar{\boldsymbol{S}}_{\theta_1^{(0)}}(\boldsymbol{x})$ given according to (11), i.e.,

$$\theta_2^{(0)} = \arg\min_{\theta_2}\sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}}\|\boldsymbol{Q}_{\theta_2}(\boldsymbol{x}_i) - \bar{\boldsymbol{S}}_{\theta_1^{(0)}}(\boldsymbol{x})\|^2. \tag{14}$$

The initialization step is important since if the initial guess is close to a global optimum, the second step can lead right to it. Note that the choice of the least-square criterion is mainly due

[2] Prior information such as the sparsity, when available, can be added to improve the estimation.

to the fact that it can be solved efficiently. In particular, for the linear self interference model, where $\theta_1 = \boldsymbol{A}$ and $\theta_2 = (\boldsymbol{B}, \boldsymbol{C})$, we propose the following closed-form initialization.

Proposition 1. *The least-square initialization of $\boldsymbol{A}$ is given by*

$$\boldsymbol{A}^{(0)} = \left(\sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \bar{\boldsymbol{y}}_{\boldsymbol{x}} \boldsymbol{x}^\intercal\right)\left(\sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \boldsymbol{x}\boldsymbol{x}^\intercal\right)^{-1}. \quad (15)$$

An approximate least-square initialization of $\boldsymbol{B}$ and $\boldsymbol{C}$ is the Cholesky decomposition of $(\Xi)_+$ and $(\boldsymbol{G})_+$, respectively, where $\left[\begin{smallmatrix}\Xi\\ \boldsymbol{G}\end{smallmatrix}\right] = \mathrm{perm}^{-1}(\boldsymbol{R}^\dagger \boldsymbol{V})$ with

$$\boldsymbol{R} := \begin{bmatrix} 1 & \sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \boldsymbol{x}^\intercal \otimes \boldsymbol{x}^\intercal \\ \sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \boldsymbol{x} \otimes \boldsymbol{x} & \sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \boldsymbol{x}\boldsymbol{x}^\intercal \otimes \boldsymbol{x}\boldsymbol{x}^\intercal \end{bmatrix}, \quad (16)$$

$$\boldsymbol{V} := \mathrm{perm}\left(\sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \hat{\boldsymbol{Q}}_{\boldsymbol{x}}, \sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \boldsymbol{D}_{\boldsymbol{x}}^\intercal \hat{\boldsymbol{Q}}_{\boldsymbol{x}} \boldsymbol{D}_{\boldsymbol{x}}\right), \quad (17)$$

and

$$\hat{\boldsymbol{Q}}_{\boldsymbol{x}} := \frac{1}{T\pi_{\boldsymbol{x}}} \sum_{i\in[T]:\, \boldsymbol{x}_i = \boldsymbol{x}} (\boldsymbol{y}_i - \boldsymbol{A}^{(0)}\boldsymbol{x})(\boldsymbol{y}_i - \boldsymbol{A}^{(0)}\boldsymbol{x})^\intercal, \quad (18)$$

$\boldsymbol{D}_{\boldsymbol{x}} := \mathbf{I}_N \otimes \boldsymbol{x}^\intercal$, and the mapping $\mathrm{perm}(\cdot,\cdot)$ is defined in Appendix A.

Proof. See Appendix A. □

Note that the proposed initial value of $(\boldsymbol{B}, \boldsymbol{C})$ is not the exact solution of (14) in general, unless Ξ and $\boldsymbol{G}$ happen to be positive semi-definite (they are both symmetric though). Nevertheless, it provides a simple and reasonable initialization for the original ML problem.

2) Step 2: Gradient Descent: In this step, we apply gradient decent starting from the initialized values obtained Step 1. The first-order differential of $L(\theta)$ is given by

$$\mathrm{d}L = \sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \mathrm{d}\left(f\left(\boldsymbol{Y}_{\boldsymbol{x}} - \boldsymbol{\mu}_{\theta_1}(\boldsymbol{x})\mathbf{1}^\intercal, \boldsymbol{Q}_{\theta_2}(\boldsymbol{x})\right)\right). \quad (19)$$

Applying the matrix differential [8]

$$\mathrm{d}f(\boldsymbol{M}, \boldsymbol{Q}) = 2\mathrm{tr}(\boldsymbol{M}^\intercal \boldsymbol{Q}^{-1} \mathrm{d}\boldsymbol{M}) + \mathrm{tr}((\boldsymbol{Q}^{-1} - \boldsymbol{Q}^{-1}\boldsymbol{M}\boldsymbol{M}^\intercal \boldsymbol{Q}^{-1})\mathrm{d}\boldsymbol{Q}), \quad (20)$$

and the chain rule, we have

$$\begin{aligned}\mathrm{d}L = &-2\sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} (\bar{\boldsymbol{y}}_{\boldsymbol{x}} - \boldsymbol{x})^\intercal \boldsymbol{Q}_{\theta_2}^{-1}(\boldsymbol{x}) \mathrm{d}\boldsymbol{\mu}_{\theta_1}(\boldsymbol{x}) \\ &+2\sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \mathrm{tr}\Big(\boldsymbol{B}_{\theta_2}^\intercal (\boldsymbol{Q}_{\theta_2}^{-1}(\boldsymbol{x})(\boldsymbol{Q}_{\theta_2}(\boldsymbol{x}) - \bar{\boldsymbol{S}}_{\theta_1}(\boldsymbol{x}))\boldsymbol{Q}_{\theta_2}^{-1}(\boldsymbol{x}))\mathrm{d}\boldsymbol{B}_{\theta_2} \\ &+\boldsymbol{W}_{\theta_2}(\boldsymbol{x})^\intercal \boldsymbol{Q}_{\theta_2}^{-1}(\boldsymbol{x})(\boldsymbol{Q}_{\theta_2}(\boldsymbol{x}) - \bar{\boldsymbol{S}}_{\theta_1}(\boldsymbol{x}))\boldsymbol{Q}_{\theta_2}^{-1}(\boldsymbol{x})\mathrm{d}\boldsymbol{W}_{\theta_2}(\boldsymbol{x})\Big). \quad (21)\end{aligned}$$

We propose to optimize θ_1 and θ_2 in an iterative manner as follows. Each iteration i, $i = 1, 2, \dots$, contains two steps. In the first step, we fix $\theta_2 = \theta_2^{(i-1)}$ and minimizing (9) over θ_1. It can be shown that this is a least-square problem. In particular, when $\boldsymbol{\mu}_{\theta_1}$ is linear in θ_1, it is a linear least-square problem, and the solution is in closed form. For instance, if $\boldsymbol{\mu}_{\theta_1}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ with $\theta_1 = \boldsymbol{A}$ as in the linear self interference model, then the optimal $\boldsymbol{A}$ for a given θ_2 is such that

$$\mathrm{vec}(\boldsymbol{A}) = \left(\sum_{\boldsymbol{x}\in\mathcal{X}} (\boldsymbol{x}\boldsymbol{x}^\intercal) \otimes \boldsymbol{Q}_{\theta_2}^{-1}(\boldsymbol{x})\right)^\dagger \quad (22)$$

$$\times\ \mathrm{vec}\left(\sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \boldsymbol{Q}_{\theta_2}^{-1}(\boldsymbol{x}) \bar{\boldsymbol{y}}_{\boldsymbol{x}} \boldsymbol{x}^\intercal\right). \quad (23)$$

In the second step, we optimize θ_2 while fixing $\theta_1 = \theta_1^{(i)}$ found in the previous step. This can be done using the gradient direction that can be obtained from (21) or Newton-like (e.g., Hessian, diagonal Hessian) descent directions, together with a simple line search. In particular, in the linear self interference model ($\theta_2^{(i)} = (\boldsymbol{B}^{(i)}, \boldsymbol{C}^{(i)})$), we propose to first optimize over $\boldsymbol{B}$ for a given $\boldsymbol{C} = \boldsymbol{C}^{(i-1)}$, then optimize over $\boldsymbol{C}$ for the given $\boldsymbol{B}^{(i)}$. The iteration goes on until convergence or time limit.

IV. Receiver Design

The second problem is how to decode the information in a given GGM. In this section, we consider the detection problem, i.e., find a $\hat{\boldsymbol{x}} \in \mathcal{X}$ from the output vector $\boldsymbol{y}$, such that $\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}\in\mathcal{X}} d(\boldsymbol{x}, \boldsymbol{y})$ for some metric d. We assume that the input $\boldsymbol{x}$ is uniformly distributed over $\mathcal{X}$.

The optimal detector in this case is the ML detector, i.e., $\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg\max_{\boldsymbol{x}\in\mathcal{X}} p_\theta(\boldsymbol{y}\,|\,\boldsymbol{x}) = \arg\min_{\boldsymbol{x}\in\mathcal{X}} d_{\mathrm{ML}}(\boldsymbol{x}, \boldsymbol{y})$ where the metric is defined, through the function f in (8), as

$$d_{\mathrm{ML}}(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y} - \boldsymbol{\mu}_\theta(\boldsymbol{x}), \boldsymbol{Q}_\theta(\boldsymbol{x})). \quad (24)$$

In the conventional Gaussian channel where $\boldsymbol{Q}_\theta(\boldsymbol{x}) = \boldsymbol{Q}$ and $\boldsymbol{\mu}_\theta(\boldsymbol{x}) = \boldsymbol{H}\boldsymbol{x}$, the metric is equivalent to $\|\boldsymbol{Q}^{-\frac{1}{2}}(\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x})\|^2$, and the problem can be solved with nearest-neighbour decoding (NND) for which efficient implementations such as the sphere decoder [9] and lattice-reduction based approaches exist (see, e.g., [10]). In general, though, the metric d_{ML} depends on $\boldsymbol{x}$ in a complex and non-convex way.

One possible simplification is to relax the discrete constraint and search for the optimal solution $\boldsymbol{x}^*(\boldsymbol{y}) = \arg\min_{\boldsymbol{x}\in\mathbb{R}^M} d_{\mathrm{ML}}(\boldsymbol{x}, \boldsymbol{y})$. The solution $\boldsymbol{x}^*$ is then projected onto the discrete alphabet $\mathcal{X}$. We call this method the non-linear zero-forcing (NLZF) due to its similarity to the zero-forcing scheme in the convention model. Equivalently, the metric is

$$d_{\mathrm{NLZF}}(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{x}^*(\boldsymbol{y})\|^2. \quad (25)$$

Note that finding the optimal $\boldsymbol{x}^*(\boldsymbol{y})$ is hard since d_{ML} is not convex in $\boldsymbol{x}$ in general. So a practical implementation of the NLZF is to use gradient methods.[3]

Another detection method is inspired by the previous observation that fixing the covariance matrix $\boldsymbol{Q}_\theta(\boldsymbol{x}) = \boldsymbol{Q}$, the objective function becomes $f(\boldsymbol{y} - \boldsymbol{\mu}_\theta(\boldsymbol{x}), \boldsymbol{Q})$ which can be solved with NND. Specifically, we first use a low-complexity method to obtain a rough estimate $\boldsymbol{x}_0(\boldsymbol{y})$, then we search for $\boldsymbol{x}$ that minimizes $f(\boldsymbol{y} - \boldsymbol{\mu}_\theta(\boldsymbol{x}), \boldsymbol{Q}_\theta(\boldsymbol{x}_0(\boldsymbol{y})))$. This two-step procedure is called *self-interference whitening*. It was first

[3]The gradient can be derived from (20) and the chain rule of derivatives.

proposed for the discrete-time MIMO phase noise channel in [4]. Equivalently, the decoding metric is

$$d_{\text{SIW}}(\boldsymbol{x},\boldsymbol{y}) = f(\boldsymbol{y}-\boldsymbol{\mu}_\theta(\boldsymbol{x}), \boldsymbol{Q}_\theta(\boldsymbol{x}_0(\boldsymbol{y}))). \quad (26)$$

One can improve the performance of SIW with iterations. Indeed, $\boldsymbol{x}_0$ can be updated each time we find an estimate with SIW. Then, we repeat the SIW with the new $\boldsymbol{x}_0$. If the solution of the SIW gives a lower value of d_{ML}, then it is retained and we continue with a new iteration.

Finally, we can perform a simple linear minimum mean square error (LMMSE) estimator, namely, $\boldsymbol{x}_{\text{LMMSE}} = \boldsymbol{F}\boldsymbol{y}$ where

$$\boldsymbol{F} := \mathbb{E}\left[\boldsymbol{x}\boldsymbol{y}^\intercal\right]\mathbb{E}\left[\boldsymbol{y}\boldsymbol{y}^\intercal\right]^{-1} \quad (27)$$

$$= \mathbb{E}\left[\boldsymbol{x}\boldsymbol{\mu}_\theta(\boldsymbol{x})^\intercal\right]\mathbb{E}\left[\boldsymbol{\mu}_\theta(\boldsymbol{x})\boldsymbol{\mu}_\theta(\boldsymbol{x})^\intercal + \boldsymbol{Q}_\theta(\boldsymbol{x})\right]^{-1}. \quad (28)$$

The decoding metric is

$$d_{\text{LMMSE}}(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{F}\boldsymbol{y}\|^2. \quad (29)$$

V. Capacity Analysis

The capacity of a discrete-time memoryless channel with input $\boldsymbol{x}$ and output $\boldsymbol{y}$ is $C = \sup_{P_X} I(\boldsymbol{x};\boldsymbol{y})$ where the supremum is subject to any input constraint. Finding a closed-form expression of the channel capacity remains an open problem for most channels. We are interested in finding numerically evaluable capacity upper and lower bounds under the average power constraint $\mathbb{E}\left[\|\boldsymbol{x}\|^2\right] \le P$.

For a given input distribution P_X over an alphabet $\mathcal{X}$, the mutual information is

$$I(\boldsymbol{x};\boldsymbol{y}) = \mathbb{E}\left[\log\frac{p_\theta(\boldsymbol{y}\,|\,\boldsymbol{x})}{\mathbb{E}_{\boldsymbol{x}'\sim P_X}\left[p_\theta(\boldsymbol{y}\,|\,\boldsymbol{x}')\right]}\right], \quad (30)$$

which can be evaluated numerically. The mutual information is an achievable rate with an i.i.d. codebook and optimal decoding. With the same codebook but an arbitrary decoding metric $d(\boldsymbol{x},\boldsymbol{y})$, the generalized mutual information (GMI), defined as

$$I_{\text{GMI}}(\boldsymbol{x};\boldsymbol{y}) = \sup_{s>0}\mathbb{E}\left[\log\frac{e^{-sd(\boldsymbol{x},\boldsymbol{y})}}{\mathbb{E}_{\boldsymbol{x}'\sim P_X}\left[e^{-sd(\boldsymbol{x}',\boldsymbol{y})}\right]}\right], \quad (31)$$

is achievable as a lower bound of the mismatched rate (see, e.g., [6]). Here, $d(\boldsymbol{x},\boldsymbol{y})$ can be any of the metrics proposed in the previous section. Note that $I_{\text{GMI}}(\boldsymbol{x};\boldsymbol{y}) \le I(\boldsymbol{x};\boldsymbol{y})$ and the equality is achieved when $d(\boldsymbol{x},\boldsymbol{y}) = d_{\text{ML}}(\boldsymbol{x},\boldsymbol{y})$. In general, we can evaluate the GMI numerically, since a suboptimal choice of s still provides a lower bound.

For the upper bound, we focus on the linear self interference model with $\boldsymbol{B} = \mathbf{I}$ for simplicity. In particular, we follow the techniques from [7], with the so-called duality bound using the regularized Gamma distribution.

Proposition 2. *The capacity of the linear self interference model (see Definition 1) with $\boldsymbol{B}_\theta = \mathbf{I}$ is upper-bounded by*

$$\frac{1}{2}\sup_{\substack{\boldsymbol{x}_0\in\mathbb{R}^M\\ \|\boldsymbol{x}_0\|=1}}\sum_{i=1}^{r}\log\max\left\{N, \frac{\lambda_{\max}\left(\boldsymbol{A}^\intercal\boldsymbol{A}+\sum_{i=1}^N\boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)}{\lambda_i\left(\boldsymbol{W}(\boldsymbol{x}_0)\boldsymbol{W}(\boldsymbol{x}_0)^\intercal\right)}\right\}$$

$$+\frac{N-r}{2}\log\left(N+P\lambda_{\max}\left(\boldsymbol{A}^\intercal\boldsymbol{A}+\sum_{i=1}^N\boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)\right)$$

$$+\inf_{\alpha>0}\left(\alpha\log\left(N+P\lambda_{\max}\left(\boldsymbol{A}^\intercal\boldsymbol{A}+\sum_{i=1}^N\boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)\right)\right.$$

$$\left.+\alpha\psi(N/2)\log(e)+\log\Gamma(\alpha)-\alpha\log\alpha\right)$$

$$-\log\Gamma(N/2)-\frac{N}{2}\log(2e), \quad (32)$$

where $r := \min_{\boldsymbol{x}\in\mathbb{R}^M,\|\boldsymbol{x}\|=1}\operatorname{rank}(\boldsymbol{W}(\boldsymbol{x}))$, the matrices $\boldsymbol{C}_i \in \mathbb{R}^{M\times N'}$, $i\in[N]$, are submatrices of $\boldsymbol{C}$ such that $\boldsymbol{C} = [\boldsymbol{C}_1^\intercal\ \boldsymbol{C}_2^\intercal\ \ldots\ \boldsymbol{C}_N^\intercal]^\intercal$, $\psi(\cdot)$ denotes Euler's psi function, and $\lambda_i(\boldsymbol{M})$ and $\lambda_{\max}(\boldsymbol{M})$ denote the i-th eigenvalue in decreasing order and the largest eigenvalue, respectively, of a matrix $\boldsymbol{M}$.

Proof. See Appendix B. □

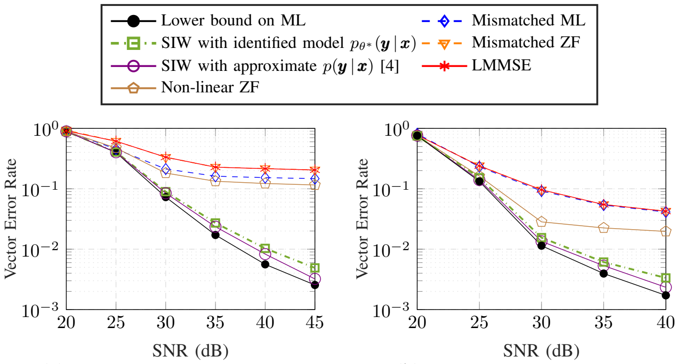
VI. Application: MIMO Phase-Noise Channel

We apply the linear self interference model to a 4×4 MIMO channel corrupted by phase noise (PN) at the receiver side. We follow the line-of-sight (LoS) model proposed in [11] with transmission distance 10 km, carrier frequency 15 GHz. The antenna spacing is set to the optimal value d_{opt} derived in [12], or only a fraction of d_{opt} due to practical constraints. Note that by reducing the antenna spacing, the condition number of the channel matrix increases.

We assume that the inputs $\boldsymbol{x}$ are drawn from M_{q}-QAM constellations. The training sequence for model identification contains 1000 replicas of 100 different 64-QAM input vectors and the corresponding output vectors. And let the identified parameter be θ^*.

In Fig. 1, we consider 256-QAM and show the vector error rate (an error is counted if the vector $\boldsymbol{x}$ is detected wrongly) of the detection schemes presented in Sec. IV under the identified model $p_{\theta^*}(\boldsymbol{y}\,|\,\boldsymbol{x})$. We also show the performance of the SIW detector under a tight approximation of the true model [4, Proposition 1], and a simulation-based lower bound of the vector error rate of the exact ML detector [4, Sec. V-A]. As can be seen, the vector error rate of the SIW detector under the identified model is only slightly higher than that under the tight approximate model, and is not far from the ML performance. For baseline, we also consider two detectors that ignore the PN: a mismatched ML that applies the NND with the estimated channel matrix $\boldsymbol{H}$, i.e., $\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}\in\mathcal{X}}\|\boldsymbol{y}-\boldsymbol{H}\boldsymbol{x}\|^2$; and a mismatched ZF consisting in a QAM demapper of $\boldsymbol{H}^\dagger\boldsymbol{y}$. We see that LMMSE is on par with mismatched ZF, while non-linear ZF outperforms both baselines.

Finally, in Fig. 2, we consider 16-QAM and plot the achievable rate with various detection schemes given by the GMI defined in (31) for the considered 4×4 LoS PN channel with antenna spacing $0.33d_{\text{opt}}$ and $0.7d_{\text{opt}}$. It is observed that the achievable rate with the SIW detector under the identified model is only marginally lower than that with ML under the identified or exact model for the whole range of of SNR. Note that the rate achieved with ML under exact model is also



(a) Antenna spacing $0.7d_{\text{opt}}$ (b) Antenna spacing d_{opt}

Fig. 1: The vector error rate of different detection schemes for a 4×4 LoS PN channel with 256-QAM input. Here, the standard deviation of the phase variations is 2°.

the mutual information with the given input distribution. The achievable rate with mismatched ZF converges slower as the SNR increases, especially for small antenna spacing. This is because mismatched ZF involves a channel inversion which becomes unstable when the channel is ill-conditioned. For both mismatched ZF and mismatched ML, when the antenna spacing is small, the achievable rate saturates at a value lower than the limit $M_{\text{q}}n_{\text{t}}$ bits/s/Hz because of a high error floor.

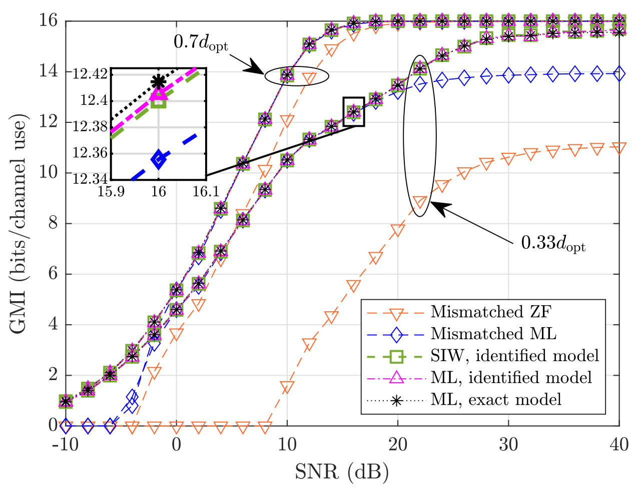


Fig. 2: The achievable rate, given by the GMI, of different detection schemes for a 4×4 LoS PN channel with 16-QAM input. Here, the standard deviation of the phase variation is 4°.

VII. Conclusion

We have proposed the *generalized Gaussian model*, a class of parametric models that can capture both non-linearities and self-interference due to multiplicative noises in wireless communications. Under this model, we investigated three key problems, namely, model identification, receiver design, and capacity analysis. We showed that this model fits closely a MIMO phase-noise channel, and the proposed signal detection algorithms perform near optimally. It would be interesting to use the GGM to address other interesting channels, e.g., with nonlinear power amplifier [13] and quantization noise [14]. Future work also includes refining and evaluating the capacity upper bound.

Appendix A
Proof of Proposition 1

Let us define $\boldsymbol{K}_{p,q}$ as the permutation matrix such that $\boldsymbol{K}_{p,q}\text{vec}(\boldsymbol{A}) = \text{vec}(\boldsymbol{A}^{\intercal})$ for any $\boldsymbol{A} \in \mathbb{R}^{p\times q}$. Let $\text{mat}_{m,n}(\boldsymbol{A})$ be a matrix whose i-th column contains $((i-1)m+1)$-th to im-th elements of $\text{vec}(\boldsymbol{A})$, $i \in [n]$. We define a mapping $\text{perm}(\cdot,\cdot) : \mathbb{R}^{M\times M} \times \mathbb{R}^{MN\times MN} \to \mathbb{R}^{(1+N^2)\times M^2}$ as

$$\text{perm}(\Xi, \boldsymbol{G}) := \begin{bmatrix} \text{vec}(\boldsymbol{X})^{\intercal} \\ \text{mat}_{N^2,M^2}(\boldsymbol{K}_{MN^2,N}\text{mat}_{MN^2,M}(\boldsymbol{G})) \end{bmatrix}, \quad (33)$$

which can be implemented by simple index permutation.

Substituting (3) into (13), after a simple manipulation, we get that the initialization of $\boldsymbol{A}$ is such that $\|\bar{\boldsymbol{Y}}\boldsymbol{\Delta}^{1/2} - \boldsymbol{A}\underline{\boldsymbol{X}}\boldsymbol{\Delta}^{1/2}\|^2$ is minimized, where $\underline{\boldsymbol{X}}$ contains the distinct elements of $\{\boldsymbol{x}_i\}_{i=1}^{T}$, $\boldsymbol{\Delta}$ contain the corresponding sample ratios in the diagonal, and $\bar{\boldsymbol{Y}}$ contains the corresponding sample means. Then, (15) follows readily since the solution to this least-square problem is well known in closed form.

Once we have $\boldsymbol{A}^{(0)}$, we can obtain an estimate of the covariance matrix $\boldsymbol{Q}(\boldsymbol{x})$ for each $\boldsymbol{x}$ given by $\hat{\boldsymbol{Q}}_{\boldsymbol{x}}$ in (18). Denote $\boldsymbol{D}_{\boldsymbol{x}} := (\mathbf{I}_N \otimes \boldsymbol{x}^{\intercal})$ and $\boldsymbol{G} := \boldsymbol{C}\boldsymbol{C}^{\intercal}$. Substituting $\boldsymbol{Q}(\boldsymbol{x}) = \Xi + \boldsymbol{W}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})^{\intercal} = \Xi + \boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{G}\boldsymbol{D}_{\boldsymbol{x}}^{\intercal}$ into (14), we get that the least-square estimates of Ξ and $\boldsymbol{G}$ are the minimizer of

$$\sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \|\Xi + \boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{G}\boldsymbol{D}_{\boldsymbol{x}}^{\intercal} - \hat{\boldsymbol{Q}}_{\boldsymbol{x}}\|^2 = \sum_{\boldsymbol{x}\in\mathcal{X}} \pi_{\boldsymbol{x}} \|\text{vec}(\Xi) + (\boldsymbol{D}_{\boldsymbol{x}} \otimes \boldsymbol{D}_{\boldsymbol{x}})\text{vec}(\boldsymbol{G}) - \text{vec}(\hat{\boldsymbol{Q}}_{\boldsymbol{x}})\|^2.$$

After some manipulations, the minimizer is such that

$$\begin{bmatrix} T\mathbf{I}_{M^2} & \sum_{i=1}^{T} \boldsymbol{D}_{\boldsymbol{x}_i} \otimes \boldsymbol{D}_{\boldsymbol{x}_i} \\ \sum_{i=1}^{T} \boldsymbol{D}_{\boldsymbol{x}_i}^{T} \otimes \boldsymbol{D}_{\boldsymbol{x}_i}^{T} & \sum_{i=1}^{T} \boldsymbol{D}_{\boldsymbol{x}_i}^{\intercal}\boldsymbol{D}_{\boldsymbol{x}_i} \otimes \boldsymbol{D}_{\boldsymbol{x}_i}^{\intercal}\boldsymbol{D}_{\boldsymbol{x}_i} \end{bmatrix} \begin{bmatrix} \text{vec}(\Xi) \\ \text{vec}(\boldsymbol{G}) \end{bmatrix} = \begin{bmatrix} \text{vec}(\sum_{i=1}^{T} \hat{\boldsymbol{Q}}_{\boldsymbol{x}_i}) \\ \text{vec}(\sum_{i=1}^{T} \boldsymbol{D}_{\boldsymbol{x}_i}^{\intercal}\hat{\boldsymbol{Q}}_{\boldsymbol{x}_i}\boldsymbol{D}_{\boldsymbol{x}_i}) \end{bmatrix}. \quad (34)$$

Instead of solving the above linear system directly, which requires inverting a large matrix of size M^2N^2, we can show that it is equivalent to solving the following system of size N^2

$$\begin{bmatrix} T & \sum_{i=1}^{T} \boldsymbol{x}_i^{\intercal} \otimes \boldsymbol{x}_i^{\intercal} \\ \sum_{i=1}^{T} \boldsymbol{x}_i \otimes \boldsymbol{x}_i & \sum_{i=1}^{T} \boldsymbol{x}_i\boldsymbol{x}_i^{\intercal} \otimes \boldsymbol{x}_i\boldsymbol{x}_i^{\intercal} \end{bmatrix} \text{perm}(\Xi, \boldsymbol{G}) = \text{perm}\left(\sum_{i=1}^{T} \hat{\boldsymbol{Q}}_{\boldsymbol{x}_i}, \sum_{i=1}^{T} \boldsymbol{D}_{\boldsymbol{x}_i}^{\intercal}\hat{\boldsymbol{Q}}_{\boldsymbol{x}_i}\boldsymbol{D}_{\boldsymbol{x}_i}\right), \quad (35)$$

or, equivalently, $\boldsymbol{R} \cdot \text{perm}(\Xi, \boldsymbol{G}) = \boldsymbol{V}$ with $\boldsymbol{R}$ and $\boldsymbol{V}$ given in (16) and (17), respectively. Therefore, we can obtain the estimates of Ξ and $\boldsymbol{G}$ from $\begin{bmatrix} \Xi \\ \boldsymbol{G} \end{bmatrix} = \text{perm}^{-1}\left(\boldsymbol{R}^{\dagger}\boldsymbol{V}\right)$.

Appendix B
Proof of Proposition 2

In this appendix, let $\{\lambda_i(\boldsymbol{M})\}_{i=1}^{m}$ denote the eigenvalues of an $m \times m$ matrix $\boldsymbol{M}$ sorted in decreasing order, and $\lambda_{\max}(\boldsymbol{M})$

denote the largest eigenvalue of $\boldsymbol{M}$, i.e., $\lambda_{\max}(\boldsymbol{M}) = \lambda_1(\boldsymbol{M})$. Recall that the linear self interference model is equivalent to

$$\boldsymbol{y} = \boldsymbol{\mu}(\boldsymbol{x}) + \boldsymbol{W}(\boldsymbol{x})\tilde{\boldsymbol{z}} + \boldsymbol{B}\boldsymbol{z} \tag{36}$$

where $\tilde{\boldsymbol{z}}, \boldsymbol{z}$ are i.i.d. as $\mathcal{N}(0, \mathbf{I}_N)$, $\boldsymbol{B} \in \mathbb{R}^{N\times N}$, $\boldsymbol{\mu}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ with $\boldsymbol{A} \in \mathbb{R}^{N\times M}$, and $\boldsymbol{W}(\boldsymbol{x}) = (\mathbf{I}_N \otimes \boldsymbol{x}^\intercal)\boldsymbol{C}$ with $\boldsymbol{C} \in \mathbb{R}^{MN\times N'}$. Let us assume that $\boldsymbol{B} = \mathbf{I}$. (Otherwise, one can whiten $\boldsymbol{y}$ to obtain $\boldsymbol{B}^{-1}\boldsymbol{y}$.) We would like to upper-bound $C = \sup_{P_X} I(\boldsymbol{x}; \boldsymbol{y})$. To this end, we follow the duality approach [7] and proceed as

$$\begin{aligned} I(\boldsymbol{x};\boldsymbol{y}) &= h(\boldsymbol{y}) - h(\boldsymbol{y}\,|\,\boldsymbol{x}) && (37)\\ &= \mathbb{E}\left[-\log p(\boldsymbol{y})\right] - h(\boldsymbol{y}\,|\,\boldsymbol{x}) && (38)\\ &= \mathbb{E}\left[-\log q(\boldsymbol{y})\right] - D(p\|q) - h(\boldsymbol{y}\,|\,\boldsymbol{x}) && (39)\\ &\le \mathbb{E}\left[-\log q(\boldsymbol{y})\right] - h(\boldsymbol{y}\,|\,\boldsymbol{x}), && (40) \end{aligned}$$

due to the non-negativity of the KL divergence $D(p\|q)$. Here, $p(\boldsymbol{y})$ is the true pdf of $\boldsymbol{y}$ induced by the model and $q(\boldsymbol{y})$ is an arbitrary pdf in $\mathbb{R}^N$. By choosing $q(\boldsymbol{y})$ as the regularized Gamma distribution [7, Eq.(23)], we can show the following upper-bound

$$\begin{aligned} I(\boldsymbol{x};\boldsymbol{y}) \le{}& \log \pi^{N/2} - \log\Gamma(N/2) + \log\Gamma(\alpha) - \alpha\log\alpha \\ &+\Big(\frac{N}{2} - \alpha\Big)\mathbb{E}\left[\log\|\boldsymbol{y}\|^2\right] - h(\boldsymbol{y}\,|\,\boldsymbol{x}) \\ &+\alpha(1 + \log\mathbb{E}\left[\|\boldsymbol{y}\|^2\right]), \quad \alpha > 0. && (41) \end{aligned}$$

(This bound can be obtained by taking $\mathsf{A} = \mathbf{I}$ in [7, Eq.(27)].) Given $\boldsymbol{x}$, the output $\boldsymbol{y}$ follows the Gaussian distribution with mean $\boldsymbol{\mu}(\boldsymbol{x})$ and covariance matrix $\mathbf{I} + \boldsymbol{W}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})^\intercal$. Thus, the conditional entropy $h(\boldsymbol{y}\,|\,\boldsymbol{x})$ can be easily computed as

$$\begin{aligned} h(\boldsymbol{y}\,|\,\boldsymbol{x}) = {}& \frac{N}{2}\log(2\pi e) \\ &+\frac{1}{2}\mathbb{E}\left[\log\det(\mathbf{I} + \boldsymbol{W}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})^\intercal)\right], && (42) \end{aligned}$$

Let us define

$$r := \min_{\boldsymbol{x}\in\mathbb{R}^M, \|\boldsymbol{x}\|=1} \operatorname{rank}\left(\boldsymbol{W}(\boldsymbol{x})\right). \tag{43}$$

Note that $r \le N$. We expand the right-hand side of (41) as

$$\begin{aligned} &I(\boldsymbol{x};\boldsymbol{y}) \\ &\le \underbrace{\frac{1}{2}\mathbb{E}_{\boldsymbol{x}}\left[r\mathbb{E}_{\boldsymbol{z},\tilde{\boldsymbol{z}}}[\log\|\boldsymbol{y}\|^2] - \log\det(\mathbf{I} + \boldsymbol{W}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})^\intercal)\right]}_{c_1} \\ &\quad + \underbrace{\frac{N-r}{2}\mathbb{E}\left[\log\|\boldsymbol{y}\|^2\right]}_{c_2} \\ &\quad + \underbrace{\alpha\left(\log\mathbb{E}\left[\|\boldsymbol{y}\|^2\right] - \mathbb{E}\left[\log\|\boldsymbol{y}\|^2\right] + 1\right) + \log\Gamma(\alpha) - \alpha\log\alpha}_{c_3} \\ &\quad - \log\Gamma(N/2) - \frac{N}{2}\log(2e). && (44) \end{aligned}$$

We proceed to bound the terms c_1, c_2, and c_3.

First, by using Jensen's inequality, we have that

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{z},\tilde{\boldsymbol{z}}}[\log\|\boldsymbol{y}\|^2] \\ &\le \log\mathbb{E}_{\boldsymbol{z},\tilde{\boldsymbol{z}}}[\|\boldsymbol{y}\|^2] && (45)\\ &= \log\left(\boldsymbol{x}^\intercal\boldsymbol{A}^\intercal\boldsymbol{A}\boldsymbol{x} + \operatorname{tr}\left(\boldsymbol{W}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})^\intercal\right) + \operatorname{tr}(\mathbf{I})\right) && (46)\\ &= \log\left(N + \boldsymbol{x}^\intercal\left(\boldsymbol{A}^\intercal\boldsymbol{A} + \sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)\boldsymbol{x}\right) && (47)\\ &\le \log\left(N + \|\boldsymbol{x}\|^2\lambda_{\max}\left(\boldsymbol{A}^\intercal\boldsymbol{A} + \sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)\right). && (48) \end{aligned}$$

Here, in (47), we denote $\boldsymbol{C} = [\boldsymbol{C}_1^\intercal\ \boldsymbol{C}_2^\intercal\ \ldots\ \boldsymbol{C}_N^\intercal]^\intercal$ and use that $\operatorname{tr}\left(\boldsymbol{W}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})^\intercal\right) = \boldsymbol{x}^\intercal\left(\sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)\boldsymbol{x}$ since

$$\boldsymbol{W}(\boldsymbol{x}) = (\mathbf{I}_N \otimes \boldsymbol{x}^\intercal)\boldsymbol{C} = \begin{bmatrix} \boldsymbol{x}^\intercal\boldsymbol{C}_1 \\ \boldsymbol{x}^\intercal\boldsymbol{C}_2 \\ \vdots \\ \boldsymbol{x}^\intercal\boldsymbol{C}_N \end{bmatrix};$$

(48) follows from [15, Th. 4.2.2-c]. By using (48) and replacing the expectation over $\boldsymbol{x}$ by the supremum, the term c_1 is upper-bounded as

$$\begin{aligned} &c_1 \\ &\le \frac{1}{2}\sup_{\boldsymbol{x}\in\mathbb{R}^M}\Bigg[\log\left(N + \|\boldsymbol{x}\|^2\lambda_{\max}\left(\boldsymbol{A}^\intercal\boldsymbol{A} + \sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)\right)^r \\ &\qquad - \log\det(\mathbf{I} + \boldsymbol{W}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})^\intercal)\Bigg] && (49)\\ &= \frac{1}{2}\sup_{\boldsymbol{x}\in\mathbb{R}^M}\Bigg[\log\left(N + \|\boldsymbol{x}\|^2\lambda_{\max}\left(\boldsymbol{A}^\intercal\boldsymbol{A} + \sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)\right)^r \\ &\qquad - \sum_{i=1}^N \log\lambda_i(\mathbf{I} + \boldsymbol{W}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})^\intercal)\Bigg] && (50)\\ &\le \frac{1}{2}\sup_{\boldsymbol{x}\in\mathbb{R}^M}\Bigg[\log\left(N + \|\boldsymbol{x}\|^2\lambda_{\max}\left(\boldsymbol{A}^\intercal\boldsymbol{A} + \sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)\right)^r \\ &\qquad - \sum_{i=1}^r \log\left(1 + \|\boldsymbol{x}\|^2\lambda_i\left(\boldsymbol{W}\Big(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\Big)\boldsymbol{W}\Big(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\Big)^\intercal\right)\right)\Bigg] && (51)\\ &= \frac{1}{2}\sup_{\boldsymbol{x}\in\mathbb{R}^M}\sum_{i=1}^r \log\frac{N + \|\boldsymbol{x}\|^2\lambda_{\max}\left(\boldsymbol{A}^\intercal\boldsymbol{A} + \sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)}{1 + \|\boldsymbol{x}\|^2\lambda_i\left(\boldsymbol{W}(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|})\boldsymbol{W}(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|})^\intercal\right)} && (52)\\ &\le \frac{1}{2}\sup_{\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}}\sum_{i=1}^r \log\sup_{\|\boldsymbol{x}\|}\frac{N + \|\boldsymbol{x}\|^2\lambda_{\max}\left(\boldsymbol{A}^\intercal\boldsymbol{A} + \sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)}{1 + \|\boldsymbol{x}\|^2\lambda_i\left(\boldsymbol{W}(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|})\boldsymbol{W}(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|})^\intercal\right)}. && (53) \end{aligned}$$

Note that for any $\boldsymbol{x}_0 \in \mathbb{R}^M$, $\|\boldsymbol{x}_0\| = 1$, it holds that

$$\begin{aligned} \lambda_i\left(\boldsymbol{W}(\boldsymbol{x}_0)\boldsymbol{W}(\boldsymbol{x}_0)^\intercal\right) &\le \operatorname{tr}\left(\boldsymbol{W}(\boldsymbol{x}_0)\boldsymbol{W}(\boldsymbol{x}_0)^\intercal\right) && (54)\\ &= \boldsymbol{x}_0^\intercal\left(\sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right)\boldsymbol{x}_0 && (55)\\ &\le \|\boldsymbol{x}_0\|^2\lambda_{\max}\left(\sum_{i=1}^N \boldsymbol{C}_i\boldsymbol{C}_i^\intercal\right) && (56) \end{aligned}$$

$$\leq \lambda_{\max}\left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}+\sum_{i=1}^{N}\boldsymbol{C}_i\boldsymbol{C}_i^{\mathsf{T}}\right). \quad (57)$$

Thus, the terms inside the $\log$ in (53) have the form $\sup_{x\geq 0}\frac{N+ax}{1+bx}$ with $a \geq b$. It can be shown easily that $\sup_{x\geq 0}\frac{N+ax}{1+bx} = \max\{N, a/b\}$, where the supremum is achieved at $x=0$ if $N \geq a/b$ and $x=\infty$ if $N < a/b$. Therefore, the upper-bound of c_1 in (53) can be expressed as

$$c_1 \leq \frac{1}{2}\sup_{\substack{\boldsymbol{x}_0\in\mathbb{R}^M\\ \|\boldsymbol{x}_0\|=1}}\sum_{i=1}^{r}\log\max\left\{N, \frac{\lambda_{\max}\left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}+\sum_{i=1}^{N}\boldsymbol{C}_i\boldsymbol{C}_i^{\mathsf{T}}\right)}{\lambda_i\left(\boldsymbol{W}(\boldsymbol{x}_0)\boldsymbol{W}(\boldsymbol{x}_0)^{\mathsf{T}}\right)}\right\}. \quad (58)$$

The right-hand side can be numerically evaluated since the domain of the supremum is bounded.

The term c_2 can be bounded using Jensen's inequality as in (48) as

$$c_2 \leq \frac{N-r}{2}\log\mathbb{E}\left[\|\boldsymbol{y}\|^2\right] \quad (59)$$

$$\leq \frac{N-r}{2}\log\left(N+\mathbb{E}\left[\|\boldsymbol{x}\|^2\right]\lambda_{\max}\left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}+\sum_{i=1}^{N}\boldsymbol{C}_i\boldsymbol{C}_i^{\mathsf{T}}\right)\right) \quad (60)$$

$$\leq \frac{N-r}{2}\log\left(N+P\lambda_{\max}\left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}+\sum_{i=1}^{N}\boldsymbol{C}_i\boldsymbol{C}_i^{\mathsf{T}}\right)\right). \quad (61)$$

Finally, the term c_3 is bounded as

$$c_3 \leq \alpha\left(\log\mathbb{E}\left[\|\boldsymbol{y}\|^2\right]-\mathbb{E}\left[\log\|\boldsymbol{z}\|^2\right]+1\right)+\log\Gamma(\alpha)-\alpha\log\alpha \quad (62)$$

$$\leq \alpha\log\left(N+P\lambda_{\max}\left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}+\sum_{i=1}^{N}\boldsymbol{C}_i\boldsymbol{C}_i^{\mathsf{T}}\right)\right) + \alpha\psi(N/2)\log(e)+\log\Gamma(\alpha)-\alpha\log\alpha, \quad (63)$$

where the last inequality holds because $\mathbb{E}\left[\log\|\boldsymbol{z}\|^2\right] = \psi(N/2)\log(e)+1$ with $\psi(\cdot)$ being Euler's psi function.

From the bounds of c_1, c_2, and c_3, we deduce that

$$\begin{aligned}
&I(\boldsymbol{x};\boldsymbol{y})\\
&\leq \frac{1}{2}\sup_{\substack{\boldsymbol{x}_0\in\mathbb{R}^M\\ \|\boldsymbol{x}_0\|=1}}\sum_{i=1}^{r}\log\max\left\{N, \frac{\lambda_{\max}\left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}+\sum_{i=1}^{N}\boldsymbol{C}_i\boldsymbol{C}_i^{\mathsf{T}}\right)}{\lambda_i\left(\boldsymbol{W}\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)\boldsymbol{W}\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)^{\mathsf{T}}\right)}\right\}\\
&\quad+\frac{N-r}{2}\log\left(N+P\lambda_{\max}\left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}+\sum_{i=1}^{N}\boldsymbol{C}_i\boldsymbol{C}_i^{\mathsf{T}}\right)\right)\\
&\quad+\inf_{\alpha}\left(\alpha\log\left(N+P\lambda_{\max}\left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}+\sum_{i=1}^{N}\boldsymbol{C}_i\boldsymbol{C}_i^{\mathsf{T}}\right)\right)\right.\\
&\quad\left.+\alpha\psi(N/2)\log(e)+\log\Gamma(\alpha)-\alpha\log\alpha\right)\\
&\quad-\log\Gamma(N/2)-\frac{N}{2}\log(2e). \quad (64)
\end{aligned}$$

Since this bound holds for any input distribution, it is also an upper bound on the capacity.

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