Coincidence functions and Bartlett spectra of point processes

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Abstract

The second order statistical properties of time point processes (PP) are described by the time coincidence function (CF) and the frequency Bartlett spectrum (BS). For PPs recorded by pulses appearing at random time instants, as in photodetection experiments, the CF can be measured by various physical devices showing in particular the famous bunching effect of photons. On the other hand, for PPs recorded by the intervals between successive points (lifetimes), especially for renewal processes, there is no usual procedure for the estimation of the CF, and the aim of this paper is to describe an approach of this problem. The starting point is a mathematical relation between the CF and the set of probabilities density functions of the lifetimes of any order of the PP. As a consequence the CF can be obtained by processing the results of the multiple normalized histograms of these lifetimes. In the cases, relatively rare, where the mathematical expression of the CF is known in closed form, the correct behavior of the procedure is verified by an experimental analysis of simulated data. The method is extended in order to verify the relationship between the CF and the BS.

Index Terms

Point processes, Poisson processes, renewal processes, coincidence function measurement, Bartlett spectrum, shot noise, counting methods.

I. Introduction

Numerous physical phenomena appear in the form of events occuring at random time instants. It is for example the case in some Particle Physics experiments where the purpose is to determine the instants where particles are emitted or detected. Similarly, in Statistical Optics the study of the instants when photons are detected yields information on the structure of Electromagnetic Field (Saleh, 1978). This also appears in engineering problems and the best example is the shot noise due to the random emission of electrons in electronic systems. At a macroscopic scale random time events appear also in traffic problems of communication systems (Gelenbe and Pujolle, 1998).

The mathematical description of such phenomena makes use of the theory of stochastic point processes (PP). There are numerous books and papers concerning PPs that can be classified approximately in two groups. In the first one the emphasis is dominated by the rigorous, and in general relatively abstract, aspect of the theory, without great emphasis on its applications. A good example is (Daley, 1983). Another approach presents mathematical results in a less abstract form and is more devoted to the interpretation and the use of theoretical results. A good example is (Cox and Isham, 1980). In this paper it is this second approach which is used.

This means that the introduction of the coincidence function and the Bartlett spectrum presented in the next section cannot be considered as a complete and rigorous description of these quantities, which can be found for example in (Daley, 1983). In particular our definition of the Bartlett spectrum (BS) is quite similar to the one presented on the original paper (Bartlett, 1963), even if our notations are rather different. But those interested in a more detailed understanding of the role of the coincidences in the analysis of PPs, can consult some particular references such as (Macchi, 1971, Macchi, 1975).

The purpose of this paper is to introduce a method for estimation of the CF of PPs when they defined from the distances between successive points, which is then especially well adapted to the case or renewal PPs. In order to explain the method, let us first recall the present situation concerning CF estimation or measurements. It is well known that there are two possible approaches for the statistical description or for the generation of PPs. In the first one, called counting approach, a PP is defined by the joint probability distribution of the numbers N_i of points appearing in non-overlapping arbitrary intervals ΔT_i, $1 \leq i \leq K$, where K is also an arbitrary integer. This procedure is especially well adapted to Poisson processes for which the random variables (RV) N_i are independent and with Poisson distributions defined by their mean values m_i. The second approach is called lifetime description. The lifetime of order 1 of a time PP is the distance between successive points. Let T_i denote the increasing sequence of random time instants of a PP, and let $X_i = T_i - T_{i-1}$ be the corresponding sequence of lifetimes. It is clearly a sequence of non-negative RVs and there is a one to one correspondence between T_i and X_i. As a consequence any PP is equivalent to a sequence of non-negative RVs. This approach is especially adapted to renewal stationary PPs defined by the fact that the corresponding X_is are independent an identically distributed (IID) random variables. If the mathematical equivalence of these two approaches is obvious, it is however not always easy to pass in closed form, for example, from the probability distributions of lifetimes to those of counting. The CF belongs in its definition clearly to the counting approach. Indeed the coincidence event (of order one) is defined by the fact that 2 infinitesimal distinct intervals ΔT_1 and ΔT_2 contain together 1 point of the PP. There are various electronic systems devoted to the detection of coincidences in a random sequence of pulses mathematically described as a PP. These devices are widely used in nuclear physics or in statistical optics. For example, the famous bunching effect of photons, showing that the detection of photons of an electromagnetic field can yield a PP that is not a Poisson process, is exhibited and analyzed by such coincidences experiments (Saleh, 1978). The results of these experiments are the corner stone of the validity of Quantum Mechanics in statistical optics (Picinbono and Bendjaballah, 2010). On the other hand, when a PP is defined from its lifetimes X_i, as for example all the renewal PPs, it is much more difficult to estimate the corresponding CF. This is the main purpose of the following.

This paper is then devoted to computer experiments and estimation of second-order properties of PPs. After a short review of the mathematical background and the notations used for this purpose, we present in Section III experimental results concerning the coincidence function especially in the cases where this function has an explicit mathematical expression. The last section starts from the fact that the BS which defines the second-order spectral properties of a PP is not directly measurable. However it is used in the expression of various quantities that can be easily measured. Using this remark, experimental results are presented on various different PPs and the measurements show a very good agreement between theoretical prediction and experimental result, which here also confirm the interest of the algorithms used for these experiments.

II. Fundamental Theoretical Results and Notations

Let $\mathcal{P}$ be a stationary PP of density μ. The random points of this process are time instants denoted T_i and let $N(t)$ be the random number of points T_i in the interval $[0, t[$, where 0 is an arbitrary origin of time. The function $N(t)$ is clearly a non-decreasing stepwise continuous time random function but its increments $\Delta N(t, \tau) = N(t + \tau) - N(t)$ are stationary.

It is convenient to introduce the differential increments $dN(t)$ defined by $dN(t) = \Delta N(t, dt)$. For regular PP, especially without accumulation points, these increments are Bernoulli random variables (RV) taking only the values 0 and 1, and the probability of the value 1 is μdt. As a consequence the mean value of $dN(t)$ is $\mathsf{E}[dN(t)] = \mu dt$, showing that μ is effectively the *density* of the PP. For the following discussion it is appropriate to introduce the centered increments $d\bar{N}(t)$ defined by $d\bar{N}t) = dN(t) - \mu dt$, which then are zero mean valued RVs.

Consider now two arbitrary *distinct* non-random time instants t_1 and t_2. The coincidence event at these instants can be defined by the fact that $dN(t_1)dN(t_2) = 1$. This leads to the introduction of the *coincidence function* (CF) $c(t_1, t_2)$ defined by

$$\mathsf{E}[dN(t_1)dN(t_2)] = \Pr\{[dN(t_1) = 1] \cap [dN(t_2) = 1]\} = c(t_1, t_2)dt_1 dt_2. \tag{1}$$

For regular PPs the increments $dN(t_1)$ and $dN(t_2)$ become uncorrelated when $|t_2 - t_1| \to \infty$, which implies that in this limit situation $c(t_1, t_2)$ tends to μ^2. This leads to introduce the centered CF (CCF) defined by

$$\mathsf{E}[d\bar{N}(t_1)d\bar{N}(t_2)] = \bar{c}(t_1, t_2)dt_1dt_2 = [c(t_1, t_2) - \mu^2]dt_1dt_2. \quad (2)$$

In many instances this CCF $\bar{c}(t_1, t_2)$ is easier to manipulate than the initial CF $c(t_1, t_2)$. Note, however, that these two coincidence functions require that $t_1 \neq t_2$ because the coincidence event implies that their instants are distinct. The case $t_1 = t_2$ requires a specific treatment presented later.

The CCF exhibits some obvious properties.

1. Since $\mathcal{P}$ is stationary, the CF depends only on the difference $t_1 - t_2$ and from now it is written as $c(t_1 - t_2)$. According to the previous point, $c(t)$ is defined for any nonzero time t.
2. The coincidence event is unchanged under any permutation of the instants t_1 and t_2 of (1) or (2). Consequently the CF $c(t)$ is an even (or symmetric) function of t. This property holds also for $\bar{c}(t)$.
3. It follows from the definition of $\bar{c}(t)$ that $\lim_{t\to\infty} \bar{c}(|t|) = 0$.
4. The coincidence function defined by (2) looks like a covariance function. That is, however, not the case. Indeed any covariance function must be non-negative definite, which implies a non-negative Fourier transform (FT), which is the power spectrum. We shall see later examples of coincidence functions with Fourier transforms taking negative values.

Let us now consider the case $t = 0$. For this purpose, we use the point noted above that the RV $dN(t)$ takes only the values 0 or 1. This implies that $dN(t) = dN^2(t)$. Taking into account this fact, we can extend the expression (2) which becomes for any t_1 and t_2

$$\mathsf{E}[d\bar{N}(t_1)d\bar{N}(t_2)] = [\mu\delta(t_1 - t_2) + \bar{c}(t_1 - t_2)]dt_1dt_2, \quad (3)$$

where $\delta(.)$ is the Dirac distribution. This allows us to introduce the extended CCF defined by

$$c_e(t) = \mu\delta(t) + \bar{c}(t). \quad (4)$$

Because of the term $\delta(t)$, this quantity is not an ordinary function but a distribution, even if we maintain the use of the term “function”. It has then no finite value for $t = 0$. This function is called the covariance density in the original paper of Bartlett (1963). It is then defined for any t, it is symmetric and tends to 0 when $t \to \infty$. But the basic difference from $c(t)$ is that $c_e(t)$ is now non-negative definite, as we shall see later. This implies that its FT is a non-negative function called the *Bartlett spectrum* (BS) which can be written as

$$B(\nu) = \mu + \bar{C}(\nu), \quad (5)$$

where $\bar{C}(\nu)$ is the FT of $\bar{c}(t)$.

The best and simpler examples of BS appear in the case of stationary (or homogeneois) Poisson PPs. Such processes are characterized by the fact that the increments $dN(t)$ introduced above are independent. This implies that $c(t)$ of (1) is μ^2 and $\bar{c}(t)$ of (2) is zero. This yields that the BS is constant and equal to the density μ. In terms of signal properties, this means that a Poisson PP is a *white noise* of constant spectral density μ. This fact was first published independently long time ago in the famous papers (Blanc-Lapierre, 1945, Rice, 1944) describing the shot noise at the end of the Second World War, in a period where scientific communication was almost impossible between Europe and the USA.

Poisson processes appear as appropriate statistical models of numerous physical systems at the microscopic and macroscopic scale as well. Indeed, they are characterized by the complete lack of memory and this assumption is often quite coherent with the structure of such systems. There are, however, cases where this assumption cannot be accepted, and the simplest statistical models following Poisson processes are stationary *renewal* PPs. Such PPs are defined by the fact that the time intervals between successive points, sometimes called lifetimes, are independent and identically distributed (IID) positive RVs defined by a probability density function (PDF) $f(t)$. This PDF is then the only quantity necessary for the complete mathematical description of a renewal PP. Note that a Poisson PP of density μ is also a renewal PP for which the life time is a positive exponential RV with the PDF $f(t) = \mu \exp(-\mu t)$.

There are renewal PPs for which the expression of the coincidence function can be obtained in closed form. This is especially the case of Erlang (2) processes, as we shall see later. In these cases, the calculation of the BS $B(\nu)$ of (5) is reduced to the one of the FT $\bar{C}(\nu)$ of the CCF $\bar{c}(t)$.

Since, however, the PDF $f(t)$ contains all the statistical properties of a renewal process, the BS must be expressed in terms of this function, even if its expression can be rather complicated. In fact that is not the case because there exists a relation between the CF and the so called *renewal function* (see p. 52 of Cox and Isham, 1980) which can easily be calculated from $f(t)$.

In order to introduce this function suppose that a point of $\mathcal{P}$ is recorded at time t_1. Let t_2 be an arbitrary time instant posterior to t_1. The quantity $f(t_2 - t_1)dt_2$ is the probability that *the first* point of $\mathcal{P}$ recorded after t_1 is in the interval $[t_2, t_2 + dt_2[$. Similarly $f_n(t_2 - t_1)dt_2$ has the same meaning, provided that *first* is replaced by nth. This event appears when the interval $[t_1 + dt_1, t_2[$ contains $n - 1$ points of $\mathcal{P}$. The coincidence event used for the definition of the CF by (1) appears when there are two points of $\mathcal{P}$ simultaneously in $[t_1, t_1 + dt_1[$ and $[t_2, t_2 + dt_2[$, regardless of the number of points recorded in $[t_1 + dt_1, t_2[$. This yields that $c(t_2 - t_1)$ of (1) is equal to $\mu \sum_{n=1}^{\infty} f_n(t_2 - t_1)$, where f_1 is simply f.

Let X_i, $(i \geq 1)$, be the distances between the successive points of $\mathcal{P}$ (lifetimes) posterior to t_1. It is clear that $f(t)$ and $f_n(t)$ are the PDFs of X_1 and $X_1 + X_2 + \cdots + X_n$ respectively. In the case of renewal processes this expression takes a very interesting form because the intervals X_i introduced just above are

IID random variables with the PDF $f(t)$. This implies that the PDF of their sum is the nth convolution f^{*n} of $f(t)$. As a consequence, we have $c(t) = \mu h(|t|)$ where $h(t)$ is the renewal function defined by the basic renewal equation

$$h(t) = \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} f^{*n}(t). \tag{6}$$

Note also that, since the lifetimes are positive RVs, the PDF $f(t)$ is causal, which means equal to 0 for $t < 0$. This property is valid for all the convolutions appearing in the series defining $h(t)$. The existence of a series of convolutions leads to the use of the Laplace transforms which, by the causality property, are one-sided. Let $F(s)$ be the Laplace transform of $f(t)$ defined by

$$F(s) = \int_0^{\infty} f(t)\exp(-st)dt, \tag{7}$$

which implies the property $|F(s)| < 1$. Noting that the Laplace transform of $f^{*n}(t)$ is $F(s)^n$, we deduce that (6) can be written in terms of $F(s)$ in such a way that the Laplace transform $H(s)$ of $h(t)$ becomes

$$H(s) = \frac{F(s)}{1 - F(s)}\ . \tag{8}$$

From these expressions it is possible to deduce an interesting form of the BS. From (2) we deduce that $\bar{c}(t) = c(t) - \mu^2 = \mu h(|t|) - \mu^2$ and its one-sided Laplace transform is $\mu H(s) - \mu^2/s$. As a result, the Fourier transform $\bar{C}(\nu)$ of the symmetric function $\bar{c}(t)$ is $2\Re\left[\mu H(j\omega) + \mu^2/(j\omega)\right]$, with $\omega = 2\pi\nu$, and according to (5) the expression of the BS becomes

$$B(\nu) = \mu \left[1 + 2\,\Re\left\{\frac{F(j2\pi\nu)}{1 - F(j2\pi\nu)}\right\}\right]\ . \tag{9}$$

It is interesting to verify that the BS of a stationary Poisson PP is effectively μ because in this case the term $F/(1-F)$ is purely imaginary, and then its real part is zero.

III. Estimation of Coincidence Functions

The principle of the estimation of the CF by using the relation $c(t) = \mu h(|t|)$ and Eq. (6) is based on the idea to approximate the series (6) by a finite sum of P terms, each of them being a PDF that can be estimated from observed data. The choice of the value of P can be modified by taking into account the precision of the results. An immediate consequence will however appear. It was shown above that the coincidence function tends to μ^2 when $t \to \infty$. This is obviously impossible if the series is replaced by a finite sum. Indeed each member of this sum is a PDF tending to zero for $t \to \infty$.

The first point is to know in which form the PP analyzed is recorded. As indicated before, a PP can be defined by its lifetimes which are the random distances X_i between successive points. In other words, recording a PP is equivalent to recording a set of positive RVs. Since various statistical analyses will be realized with these RVs, it is obvious that the accuracy of the results will depend on the number N of these variables. This number must be as large as possible, taking into account the possible duration of the following experiments or the memory capacity of the computers used in this task. In the experimental results presented below N is of the order of 10^7. It can be greater if necessary.

The observations X_i can come from physical experiments, as in measurement of photon coincidence in statistical optics, or be generated by computer simulation. It is this last method that is used in all what follows. This means that the simulation of a renewal PP is equivalent to the generation of a sequence of RVs that are IID with a given PDF. There are excellent programs to generate sequences of RVs W_i that are IID and uniformly distributed in the interval $[0, 1]$. By using an instantaneous appropriate non-linearity such as $X_i = f(W_i)$ it is possible to transform these W_is into X_is with a given PDF (see Devroye, 1986), and it is this procedure that we shall use to generate various renewal PPs.

In order to estimate the coincidence function, the starting point is the relation

$$c(t) = \mu \sum_{k=1}^{\infty} f_k(|t|) \tag{10}$$

indicated in the previous section. By an appropriate algorithm we shall estimate from the observation of the N RVs X_i the PDFs $f_k(t)$ and replace the series (10) by a finite sum of P terms. The choice of the value of P can be modified by taking into account the precision of the results. At this step it remains to estimate from the simulated data the P PDFs appearing in (10). This can be done easily by using P normalized histograms. But this possible procedure takes long in computing, and we shall introduce another way requiring only one histogram, which yields immediately the estimation of the coincidence function. Let us now explain its principle.

Suppose that, instead of N, we start from $N + P$ observed values X_i. From these values we construct P row $(1 \times N)$ vectors $\mathbf{S}_k$, called sum vectors and defined for $1 \leq k \leq P$ by

$$\mathbf{S}_k = \left[\sum_{1}^{k} X_i, \sum_{2}^{k+1} X_i, \ldots, \sum_{N}^{k+N-1} X_i \right] \tag{11}$$

The components of these vectors are obviously correlated and represent the distances between points associated to the life time of order k, in such a way that the marginal PDFs of these components are $f_k(t)$. As a consequence, a normalized histogram of $\mathbf{S}_k$, noted nhist $(\mathbf{S}_k)$, yields an estimation of $f_k(t)$, and the behavior of this PDF estimator is well documented in standard books of statistics.

Instead of estimating the P PDFs appearing in (10) when it contains only P terms, we shall introduce a procedure using only one histogram. For this, let $\mathbf{Y}$ be the $(1 \times N\,P)$ row vector defined by concatenation of the P vectors $\mathbf{S}_k$ defined by (12) or

$$\mathbf{Y} = [\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_P]. \tag{12}$$

In order to estimate the coincidence function by an approximation of (10) with P terms, we shall use various forms of histograms of RVs. Let us then briefly recall the principle of the PDF estimation with normalized histograms used in the following. Consider a random vector $\mathbf{X}$ with N non-negative components. A particular realization of this random vector yields a sequence of N non-negative numbers x_i which are the input signal of an histogram analysis. For this, we introduce an arbitrary number a such all the x_i satisfy $x_i < a$, which is always possible since the number N of x_is is finite. This allows us to introduce the interval analysis $[0, a]$ which is the range of possible values of the present observation of $\mathbf{X}$. This interval is divided in m adjacent and non-overlapping segments (sometimes called bins) of equal size s, which yields $m = a/s$. We then obtain a partition of $[0, a]$ in m adjacent segments. The function of the histogram is to yield the numbers n_{X,s_i} of components of $\mathbf{X}$ belonging to each segment s_i of the partition. This yields a set of m numbers called histogram of $\mathbf{X}$ and noted $\mathrm{hist}[\mathbf{X}, a, s, \{n_{X,s_i}\}]$, or simply $\mathrm{hist}[\mathbf{X}, a, s]$, when no confusion is possible. The corresponding normalized histogram noted nhist is defined by

$$\mathrm{nhist}[\mathbf{X}, a, s, \{n_{X,s_i}\}] = \frac{1}{N}\mathrm{hist}[\mathbf{X}, a, s, \{n_{sX,_i}\}], \tag{13}$$

in such a way that the sum of its elements n_{X,s_i} is 1. It is well documented in statistics that, when the RVs X_i, components of $\mathbf{X}$, have the same continuous PDF $p(x)$, the value of n_{X,s_i} is an estimation of the integral of $p(x)$ on the segment s_i. As a result the normalized histogram yields an estimation of the continuous PDF approximated by a sequence of m numbers (discrete approximation by sampling). The performances of this estimator are well known and not summarized here. It is of greater interest to investigate the properties of the normalized histogram of a vector such $\mathbf{Y}$ defined by (12), and result of the concatenation of P other vectors.

For this, let us introduce the concept of sum of histograms. Consider two $1 \times N$ row random vectors $\mathbf{V}$ and $\mathbf{W}$ and let $\mathbf{U}$ be the $1 \times 2N$ vector $[\mathbf{V}, \mathbf{W}]$ obtained by their concatenation. A realization of these vectors analyzed with two corresponding histograms defined by (a, s) yield the numbers n_{V,s_i} and n_{W,s_i} for the bin s_i. It is clear that an histogram with the same parameters (a, s) of the $1 \times 2N$ vector $\mathbf{U}$ with the same data yields the numbers $n_{U,s_i} = n_{V,s_i} + n_{W,s_i}$. This operation called sum of histograms is symbolized by the relation

$$\mathrm{hist}[\mathbf{U}, a, s, \{n_{U,s_i}\}] = \mathrm{hist}[\mathbf{V}, a, s, \{n_{V,s_i}\}] + \mathrm{hist}[\mathbf{W}, a, s, \{n_{W,s_i}\}]. \tag{14}$$

The same procedure can be introduced in order to define the product of an histogram by a real positive number. The expression can obviously extend to (12), which becomes

$$\text{hist}[\mathbf{Y}] = \sum_{i=1}^{P} \text{hist}[\mathbf{S}_i], \tag{15}$$

where all the histograms have the same parameters $(a,\, s)$. The normalized histogram of $(\mathbf{Y})$ is obtained by division by NP and noting that the division by N in the rhs of this equation yields the normalized histograms of the $\mathbf{S}_i$ we get

$$\text{nhist}[\mathbf{Y}] = \frac{1}{NP}\sum_{1}^{P} \text{hist}[\mathbf{S}_i] = \frac{1}{P}\sum_{1}^{P} \frac{1}{N}\text{hist}[\mathbf{S}_i] = \frac{1}{P}\sum_{1}^{P} \text{nhist}[\mathbf{S}_i]. \tag{16}$$

Each member of the last sum is an estimation by histogram of the PDFs appearing in (10), in such a way that the corresponding estimation of the CF deduced from (10) can be written

$$\hat{c}(t) = \mu\, P\ \text{nhist}(\mathbf{Y}). \tag{17}$$

This estimation procedure using one histogram instead of P in the use of (6) is much less time consuming than the one with calculating P distinct histograms.

As indicated just before, the data used in the histograms are not independent. But as the behavior of an histogram is described by the law of large numbers, it is well known that the validity of this law does not require the statistical independence of the data analyzed.

The first and simplest example corresponds to the case of a Poisson PP. Assuming that its density μ is 1, the PDF of the distance between successive points has a one-sided PDF, or $f(t) = u(t)\exp(-t)$, where $u(.)$ is the unit causal step function. Furthermore we have seen above that the coincidence function is constant and equal to $\mu^2 = 1$. It is especially easy to generate positive exponential RVs by computer simulation and in the results appearing in Fig. 1 the number N of samples of these RVs is 10^7 and the value of P, number of terms in the truncated series (10), is $P = 10$. The parameters of the histogram are $a = 16$ and $s = 0.1$. In the Fig. 1 we present an estimation of the PDF (points) of these data and of the corresponding CF (continuous curve). The results are presented in semi-logarithmic coordinates. The continuous curves present the theoretical values of the PDF and of the CF calculated with $P = 10$. The points are the results of the normalized histograms measuring these quantities.

This figure leads to the following comments. At first we verify that the PDF of the lifetime is perfectly estimated. A small lack of precision appears for $t > 9$ where the PDF if of order of 10^{-4}. This is obviously due to the small number of samples having such values. A better precision in this domain could be easily obtained by increasing the number of samples analyzed with the price of a greater computer time. On the other hand we verify that, as expected by the theory, $c(t) = 1$, at least for $t < 5$. This means that for an exponential distribution of mean value equal to 1, the approximation of the series (10) is perfectly valid for $t \leq 5$. This fact is uniquely due to the approximation of (10) by 10 terms. We have verified that increasing P in our experiments yields an increase of the time domain of validity of the estimation of the CF. Finally, even if this appears only partially in the figure, we have verified that our simplified algorithm introduces negligible errors in the evaluation of the series (10) limited to a sum of 10 terms. Indeed, even to values of t of order of 20, the points of the histograms remain perfectly on the theoretical curve of the sum of the first P of (10) which, as expected, tends rapidly to 0.

We shall now present similar results in the case of another PP. Points processes for which the coincidence function can be expressed explicitly are rather rare. This is however the case of Erlang (2) PPs. Such PPs can be constructed from Poisson processes by thinning regularly one point over two.

More precisely let $\mathcal{P}_P$ be a stationary Poisson PP of density λ. Its random points are denoted T_n, n integer. By a regular thinning procedure we associate a sequence of random points Θ_k defined by $\Theta_k = T_{2k}$. This new sequence of points is also a stationary PP with density $\mu = \lambda/2$. This PP is called an Erlang (2) PP $\mathcal{P}_E$, and by an obvious generalization we can construct Erlang (n) PPs of density λ/n. It is very easy to express the PDF of the distance between successive points of an Erlang (2) PP because these distances are simply the sum of two IID random variables with an exponential distribution. The result is

$$f(t) = 4\mu^2 t \exp(-2\mu t), \tag{18}$$

where μ is the density of $\mathcal{P}_E$. The calculation of the CF is a bit more complicated and yields

$$c(t) = \mu^2[1 - \exp(-4\mu|t|)]. \tag{19}$$

At this step note that the FT of this expression is clearly not a positive function. This a clear evidence that, as indicated previously, the coincidence function is not a covariance function.

We can now make the same computer experiments as those presented previously for Poisson PPs. The parameters of the experiments are exactly the same: $\mu = 1, P = 10, N = 10^7$, and the results appear in Fig. 2 where theoretical and experimental values of the PDF and of the CF are presented. The PDF is accurately estimated using normalized histograms. It is the same for the CF, but only for $t < 5$. Measurements for greater values could require more than 10 terms in the approximation of the series (10) by a finite sum. The only difference from the case of Poisson PP is that the mathematical expression of the sum of the first 10 terms of the series (10) is rather complicated in such a way that we cannot verify in this case if for $t > 5$ the experimental points are located on the theoretical curve for all values of t, as it was verified for Poisson PP.

The last computer experiment is realized with a renewal PP defined by an uniform distribution $f(t)$ of its lifetime. Assuming a density $\mu = 1$ yields $f(t) = 1/2$ for $0 \leq t \leq 2$ and zero otherwise. In this case the explicit expression of the CF is rather complicated even if it is always possible to calculate, if necessary, the multiple convolution of any order of a rectangular function. Because of this lack of simple theoretical result we slightly modified the parameters of the experiment. Instead of $P = 10$ the value $P = 14$ was chosen in order to extend the domain of validity of the measurement. Since this modification causes a significant increase of the computer time the number of samples analyzed was reduced to $N = 10^5$.

The results appear in Fig. 3 in the same form as in the previous figures. The measurement of the PDF is quite correct and the experimental points are exactly on the rectangular curve of the distribution. For the coincidence function we observe the advantage of the increase of the number of terms P for the approximation of the series (10). The asymptotic value $c(t) = 1$ is valid in the interval $[4, 8]$ for t and there is a beginning of decrease in the neighborhood of 8. This confirms the validity of our simplified algorithm of measurement of the CF. Finally for $t < 2$ we observe the dominance of the almost linear increasing of $c(t)$ due to the convolution of a rectangular function yielding a triangular function $f_2(t)$.

IV. Experiments with the Bartlett Spectrum

The Bartlett spectrum (BS) cannot easily be measured. It appears however as an important part of numerous physical experiments. The results of these experiments can be considered as an indirect approach of the BS. This is especially the case for the shot noise of PPs and we shall begin by a survey of the principal results concerning this signal and used in what follows.

The shot noise is the signal obtained at the output of a linear filter when the input are random impulses arriving at the time instants T_i of a PP $\mathcal{P}$. This output can the written as (see p. 321 of Picinbono, 1993)

$$X(t) = \sum_i h(t - T_i) = \int h(t - \theta) dN(\theta), \tag{20}$$

where $h(t)$ is the impulse response (IR) defining the filter and the $dN(\theta)$s are the differential increments introduced in Section II.

It is easy to deduce from the previous relation that the power spectrum $\Gamma_X(\nu)$ of $X(t)$ can be expressed from the BS of $\mathcal{P}$ by the relation

$$\Gamma_X(\nu) = |H(\nu)|^2 B(\nu), \tag{21}$$

where $H(\nu)$ is the frequency response (FR) of the filter, FT of $h(t)$. Since this expression is valid for any filter, that is for any function $H(\nu)$, and since $\Gamma_X(\nu)$ cannot take negative values, as power spectrum of a signal, then this property is also valid for the BS, as indicated previously. It is the simplest proof of the fact that $B(\nu) \geq 0$.

Another consequence of (21) is that the measurement of the power spectrum of the shot noise with a filter of known FR is a way to reach the BS. This procedure is well adapted to filters obtained by electronic circuits and for example the fact that the BS of a Poisson PP is equal to its density μ is a well documented result appearing already in the old papers (Blanc-Lapierre, 1945) and (Rice, 1944) and widely used in Signal Processing problems. It is however difficult to use this method in computer experiments because the generation of a shot noise is in general quite difficult when the input PP is generated by a sequence of lifetimes, as in our previous experiments. It is then necessary to overcome this inconvenience by another procedure.

Our aim is then not to measure exactly the BS but to verify that some measurable first and second order properties of the output $X(t)$ depending on the BS correspond to its theoretical values obtained by a particular BS. For this purpose we start from the fact that there is, however, one shot noise that can be easily simulated in computer experiments. It is the case of the counting shot noise. Consider the linear filter defined by the IR $h(t)$ equal to 1 for $0 \leq t \leq T$ and to 0 otherwise. It is clear that the signal $X(t)$ defined by (20) is then the number $N(t)$ of points of $\mathcal{P}$ belonging to the interval $[t-T, t]$. It is obvious that $N(t)$ obtained by this procedure corresponds to a *relaxed* counting as discussed in (Picinbono, 2013). This means that the time instant $t-T$ opening the counting interval is not necessarily a point T_i of $\mathcal{P}$. If, on the other hand, we consider only the values of $N(t)$ at the times $T_i + T$, which means that the counting interval begins at the time T_i that is a point of $\mathcal{P}$, we obtain a *triggered* counting that is not considered in the following.

The difference between the two procedures is essential in the case of renewal PPs. Indeed in the second case the intervals between successive points of the PP are IID random variables, while in the first case the first interval has not the same distribution as the other ones, which changes the probability distribution of the registered counts (for this point see (Picinbono, 2013).

Applying to this signal $N(t)$ the classical expressions of the mean and variance at the output of a linear filter we obtain the mean $m_N = \mu H(0) = \mu T$, where μ is the density of $\mathcal{P}$ and the variance

$$v_N = \mathrm{var}[N(t)] = \int_{-\infty}^{\infty} |H(\nu)|^2 B(\nu) d\nu. \tag{22}$$

For the counting filter this yields

$$v_N = 4 \int_{-\infty}^{\infty} \left[\frac{\sin(\omega T/2)}{\omega}\right]^2 B(\nu) d\nu, \tag{23}$$

with $\omega = 2\pi\nu$.

Let us now apply these expressions to some specific cases. The one of the Poisson PPs is obvious. Indeed, since the BS is constant and equal to μ, the previous equations yield $m_N = v_N = \mu T$, relations well known for a Poisson distribution.

The case of the Erlang (2) PP is more interesting because all its statistical properties can be expressed in closed form from the previous equations. Since the CF is given by (19), its Fourier transform can be inserted in (23) and the BS takes then the form

$$B_E(\nu) = \mu \left[1 - \frac{8\mu^2}{\omega^2 + 16\mu^2} \right] \;,\; \omega = 2\pi\nu. \tag{24}$$

Finally, while the CF of the renewal process with a lifetime PDF equal to 1/2 in the interval $[0, 2]$ and zero otherwise (uniform distribution) represented in Fig. 3 cannot easily be written in closed form, its BS can be obtained by simple algebra from (9) and the result is

$$B_U(\nu) = \mu \left[1 + 2 \,\frac{-4(\sin^4(\omega) + \sin(2\omega)[2\omega - \sin(2\omega)]}{4(\sin^4(\omega)) + [2\omega - \sin(2\omega)]^2} \right] \;,\; \omega = 2\pi\nu. \tag{25}$$

These Bartlett spectra calculated for $\mu = 1$ are represented in Fig. 4 for positive values of the frequency ν, which is sufficient since any BS is a symmetric function of ν. As expected, these spectra tend to the asymptotic value $\mu = 1$ called the Poisson component of the BS. But the figure shows clearly that their structure are quite different and for renewal PPs with uniform distribution the corresponding BS presents damped oscillations of frequencies $1/2$. There are due to the terms $\sin(2\omega)$ apearing in the numerator of (25).

Inserting these relations in (23) yields the variance of relaxed counting in an interval of duration T called below TVBS, for theoretical variance with Bartlett spectrum.

In the case of Erlang (2) PP the probabilities p_n for obtaining n points in the same interval can also be obtained easily, which implies that we can deduce (see Picinbono, 2013) the corresponding variance denoted TVC for theoretical variance of counting. Finally, using an algorithm described in (Picinbono, 2013) and allowing the measurement of the probabilities p_n from the distances between successive points observed experimentally, we can deduce a variance called EVC for experimental variance of counting from simulated PPs. The experience is realized with an Erlang (2) PP of density $\mu = 1$, as in the experiments of the previous section. The results are displayed in Table 1 for various values of the counting interval T appearing in (23). In this table T is this counting interval, Tm is the theoretical value of the mean of the number of points in a time interval of duration T, TVBS is the theoretical value of the variance calculated from the BS by using (22) and (23), TVC is the same quantity calculated from the counting probabilities p_n, Em is the experimental value of the mean of points and EVC the experimental variance measured from the random number of points detected in the observation interval of the filter.

It is clear that the results appearing in the second column of the table corresponding to theoretical mean values does not yield significant information since the mean value of relaxed points of a PP of density μ in an interval T is μT. The comparison between columns TVBS and TVC confirms almost perfectly that the two theoretical procedures yield the same result. Finally the statistical experience realized with 10^7 samples of a simulated Erlang (2) PP and measuring the mean and variance of the relaxed number of points of this process in an interval of duration T yield results of the last two columns. They agree quite well with those calculated theoretically. This justifies the validity of the expression of the BS inserted in the calculation of the variance of the counting shot noise in terms of the Bartlett spectrum.

TABLE 1. MEAN AND VARIANCE OF COUNTING

T	Tm	TVBS	TVC	Em	EVC
1	1	0.6227	0.6227	0.9999	0.6225
2	2.0000	1.1250	1.1250	2.0003	1.1249
3	3	1.6250	1.6251	3.0000	1.6246
4	4.0000	2.1250	2.1250	4.0006	2.1248
5	5.0000	2.6250	2.6250	5.0006	2.6253
6	6.0000	3.1250	3.1251	6.0004	3.1251
10	10.0000	5.1250	5.1251	9.9997	5.1247

The same kind of results of experiments on simulated data can be obtained with renewal PPs with lifetime of points described by an uniform distribution in the interval $[0, 2]$ which yields a mean value equal to 1. The corresponding BS appears in Fig. 4, curve 2.

There is however a difference from the Erlang (2) case because, as indicated above, for uniform distribution it is very difficult to obtain in closed form the mathematical expression of the probabilities p_n of the relaxed counting. While the theoretical mean is still given by the simple expression μT, the theoretical variance however cannot be obtained in closed form. This leads to the fact that the column 4 of the Table 1 disappears in Table 2 presenting results for renewal PPS with uniform lifetime.

TABLE 2. MEAN AND VARIANCE OF COUNTING WITH UNIFORM DISTRIBUTION

T	Tm	EM	TVBS	EVC
1	1	0.9991	0.5949	0.5949
2	2	2.0001	0.8732	0.8733
3	3	3.0004	1.2243	1.2242
4	4	4.0003	1.5564	1.5566
5	5	4.9985	1.8888	1.8887
6	6	5.9991	2.2230	2.2226
10	10	10.0005	3.5672	3.5679

But the general conclusions notes with Table 1 remain valid with Table 2 : Even if the structure of the Bartlett spectra of the two process analyzed appearing in Fig. 4 are quite different, the measurements of the variances of the number of points in various intervals of different duration yields results in quite good agreement with the values of these variances calculated from the mathematical expressions of these spectra given by using (22) and (23). This is the main result of this analysis.

V. Conclusion

The purpose of this paper was to investigate some methods for estimating second order properties of point processes (PP) when these processes are obtained from a sequence of numbers equal to the time interval between successive points. In this case, the methods used by the physicists, for example in nuclear physics or in statistical optics, cannot be used. Our study was essentially devoted to the problems of coincidence functions (CF) estimation in the time domain, and Bartlett spectra (BS) measurements in the frequency domain. The starting point is a relation between the CF and the probability density functions (PDF) $f_n(t)$ of lifetimes of any order, which are the random variables equal to the time intervals between points T_i of the of the PP and the nth points of it posterior to T_i. This relation was considered as without possible application because it uses a series of an infinite number of PDFs which must be estimated uniquely from the observation of a realization of the PP. The basic idea of our method was first, to approximate the series by a sum of a finite number P of elements, but especially to process the data in such a way that, instead of using P histograms for the estimation of the PDFs, only one is used and yields directly the CF. This procedure strongly reduces the duration of the computer calculations. Since there exists some PPs for which the mathematical expression of the CF is known, a computer simulation of the lifetime of some of then was realized in order to verify the correct behavior of the CF estimation. It appears that measured and calculated values of the CF coincide with a great degree of precision. The method was also applied in cases where no mathematical expression of the CF is known. The problem of the Bartlett spectrum is rather different. Indeed, while its mathematical definition is after numerous attempts a well established result, its direct estimation or measurement remains almost impossible. The only way to verify experimentally the theoretical results is to use an indirect way, and especially to analyze the second order statistical properties of the shot noise of a PP where the BS plays a fundamental role. It is what was discussed in the last section of the paper, where the chosen shot noise used is simply the counting of the number of points of a PP in an time interval of given duration. By using an appropriate algorithm calculating the number of points in arbitrary intervals from the time intervals between successive points of various PPs with different Bartlett spectra have shown that estimated and calculated variances were quite similar.

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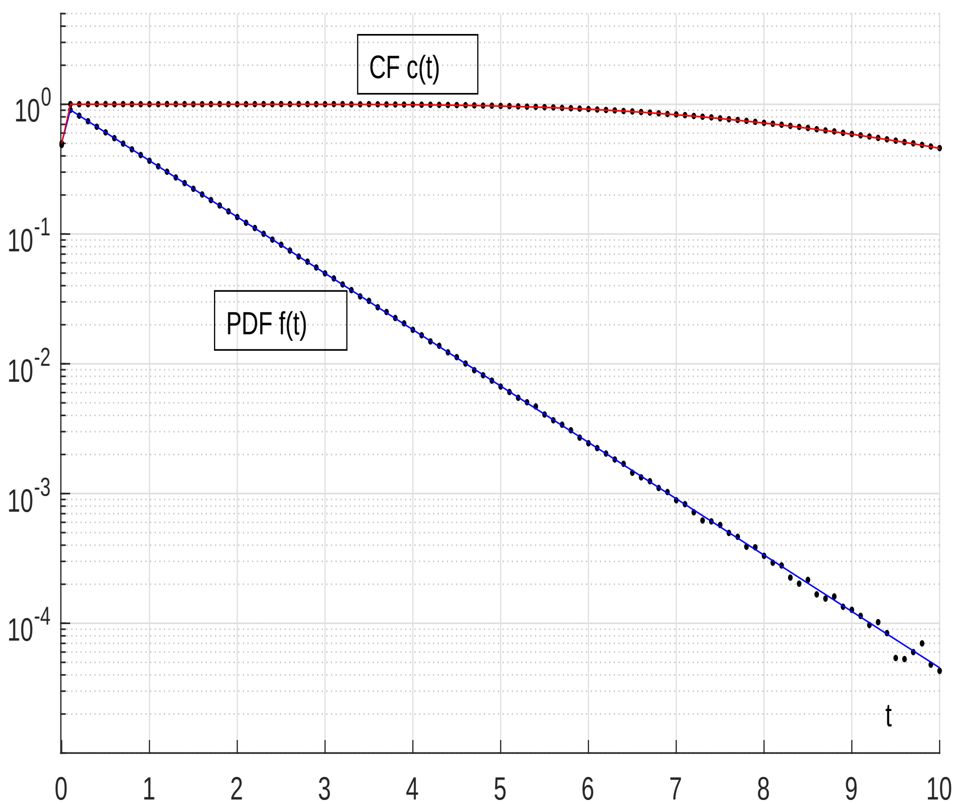

Fig. 1. Poisson Process : PDF and CF. Continuous curves : Theory ; Points : experiments.

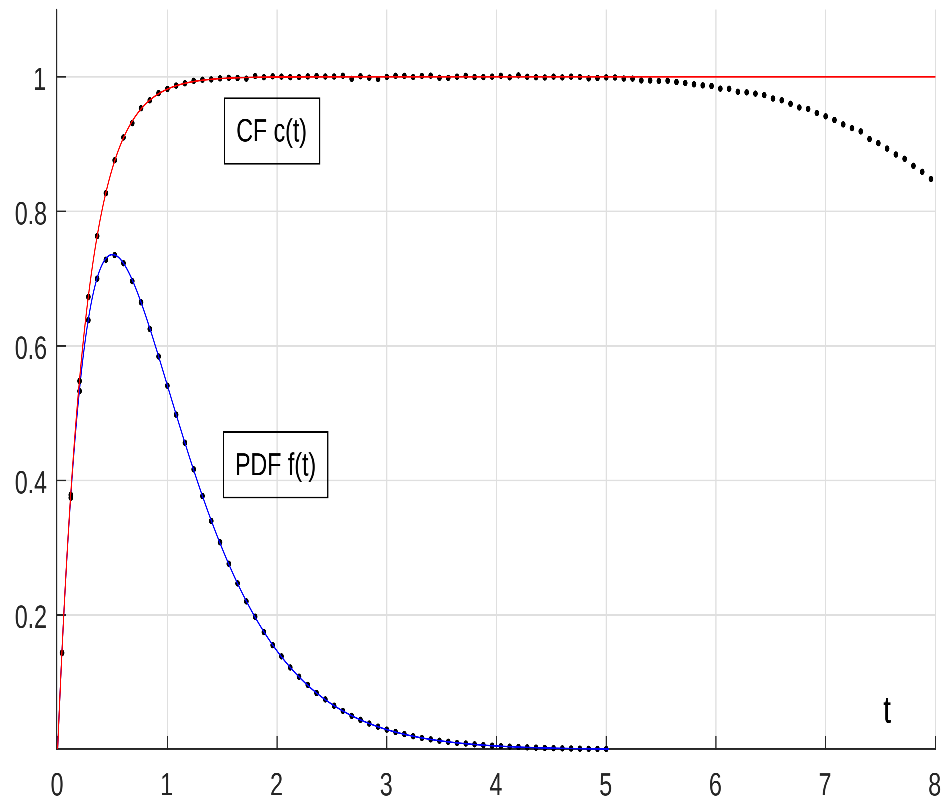

Fig. 2. Erlang (2) Process : PDF and CF. Continuous curves : Theory ; Points : experiments.

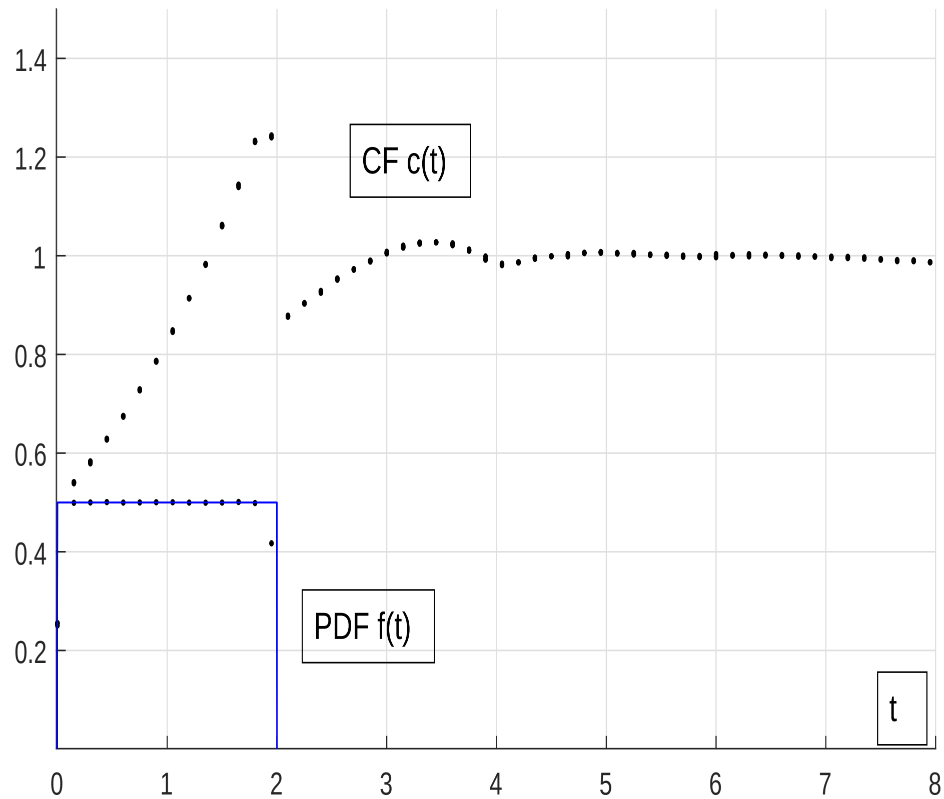

Fig. 3. Uniform Process : PDF and CF. Continuous curves : Theory ; Points : experiments.

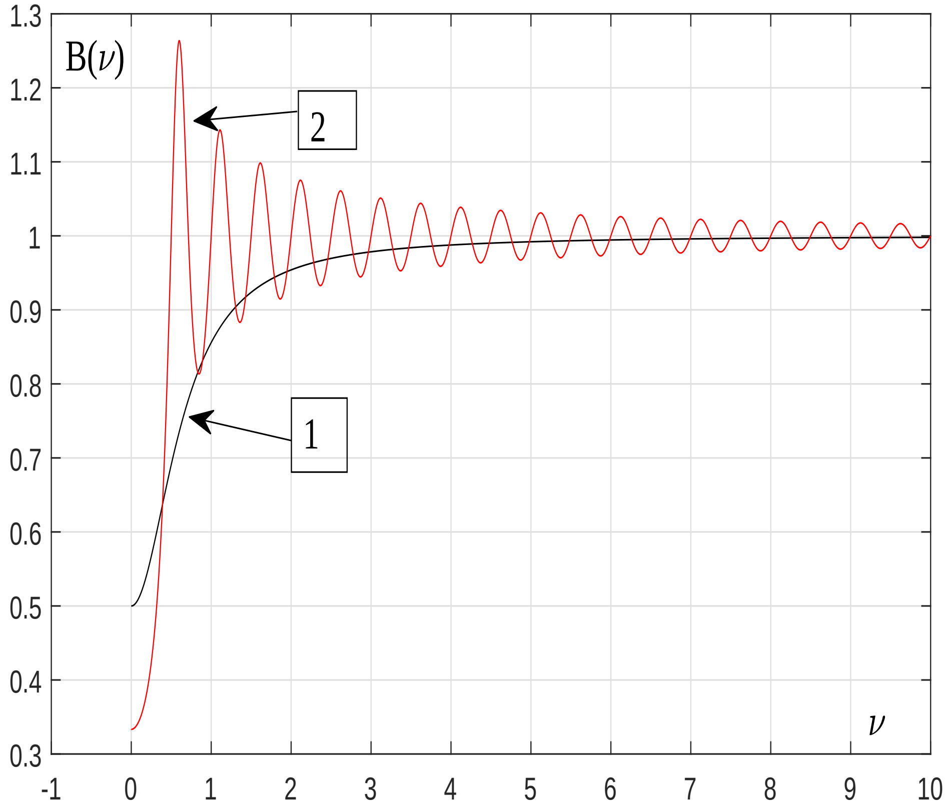

Fig. 4. Bartlett spectra for Erlang (2) process [1] and Uniform process [2].