Numerical approximation of SDEs with fractional noise and distributional drift

Ludovic Goudenège, El Mehdi Haress, Alexandre Richard

Numerical approximation of SDEs with fractional noise and distributional drift

Ludovic Goudenège* El Mehdi Haress†,‡ Alexandre Richard†,§

July 6, 2022

Abstract

We prove weak existence for multi-dimensional SDEs with distributional drift driven by a fractional Brownian motion. This holds under a condition that relates the Besov regularity of the drift to the Hurst parameter H of the noise. Then under a stronger condition, we study the numerical error between a solution X of the SDE with drift b and its Euler scheme with mollified drift b^n. We obtain a rate of convergence in $L^m(\Omega)$ for this error, which depends on the Besov regularity of the drift. This rate holds for any Hurst parameter smaller than the critical value imposed by the "strong" condition. Close to the critical value, the rate is $H-\varepsilon$. When the Besov regularity increases and the drift becomes a bounded measurable function, we recover the optimal rate of convergence $1/2-\varepsilon$.

As a byproduct of this convergence, we deduce that pathwise uniqueness holds in a class of regular Hölder continuous solutions and that any such solution is strong. The proofs rely on stochastic sewing techniques, especially to deduce new regularising properties of the discrete-time fractional Brownian motion. We also present several examples and numerical simulations that illustrate our results.

Keywords and phrases: *Numerical approximation, regularisation by noise, fractional Brownian motion.*

MSC2020 subject classification: 60H10, 65C30, 60G22, 60H50, 34A06.

1 Introduction

We are interested in the well-posedness and numerical approximation of the following d-dimensional SDE:

$$X_t = X_0 + \int_0^t b(X_s)\,ds + B_t, \quad t \in [0,1], \tag{1.1}$$

where $X_0 \in \mathbb{R}^d$, b is a distribution in some nonhomogeneous Besov space $\mathcal{B}_p^\gamma$ and B is an $\mathbb{R}^d$-fractional Brownian motion (fBm) with Hurst parameter H. When B is a standard Brownian motion ($H=1/2$), this equation received a lot of attention when the drift is irregular, see for instance [35, 37] for bounded measurable drift or [19] under some integrability condition. Strong well-posedness were obtained in those cases, which contrasts with the non-uniqueness and sometimes non-existence that can happen for the corresponding equations without noise. In the case B is a fractional Brownian motion, the results are more recent and we refer to Nualart and Ouknine [26] for Hölder continuous drifts, then to Baños et al. [4], Catellier and Gubinelli [7] and Anzeletti et al. [1] for distributional drifts when the Hurst parameter is smaller than $1/2$.

*Fédération de Mathématiques de CentraleSupélec, CNRS FR-3487; Université Paris-Saclay, France, goudenege@math.cnrs.fr. This work is supported by the SIMALIN project ANR-19-CE40-0016 from the French National Agency.

†Université Paris-Saclay, CentraleSupélec, MICS and CNRS FR-3487, France.

‡el-mehdi.haress@centralesupelec.fr. EH acknowledges the support of the Labex de Mathématiques Hadamard.

§alexandre.richard@centralesupelec.fr

The most simple approximation scheme for (1.1) is given by the Euler scheme with a time-step h

$$X_t^h = X_0 + \int_0^t b(X_{r_h}^h)\, dr + B_t, \quad t \in [0,1],$$

where $r_h = h\lfloor \frac{r}{h} \rfloor$. For the numerical analysis of Brownian SDEs with smooth coefficients, including the previous scheme and higher-order approximations, we point to a few classical works by Pardoux, Talay and Tubaro [28, 34], see also [18]. The strong error $\|X_t - X_t^h\|_{L^m(\Omega)}$ is known to be of order h (and $h^{1/2}$ when the noise is multiplicative). When the coefficients are irregular, Dareiotis et al. [8] obtained recently a strong error with the optimal rate of order 1/2 for merely bounded measurable drifts, even if the noise is multiplicative. We also refer to the review [33] and references therein for discontinuous coefficients, and the recent weak error analysis of Jourdain and Menozzi [17] for integrable drifts. Besides, we mention that when the drift is a distribution in a Bessel potential space with negative regularity, De Angelis et al. [10] have obtained a rate of convergence for the so-called virtual solutions of a (Brownian) SDE, using a 2-step mollification procedure of the drift.

Let us now recall briefly what is known when B is a fractional Brownian motion. First, Neuenkirch and Nourdin [25] considered one-dimensional equations with $H > 1/2$, smooth coefficients and multiplicative noise, i.e. the more general case with B replaced by a symmetric Russo-Vallois [32] integral $\int_0^t \sigma(X_s)\, d^o B_s$ in (1.1). They proved that the rate of convergence for the strong error is exactly of order $2H - 1$. Then Hu et al. [16] introduced a modified Euler scheme to obtain an improved convergence rate of order $2H - 1/2$, still in the multiplicative case. They also derived an interesting weak error rate of convergence. Recently, Butkovsky et al. [6] considered (1.1) with any Hurst parameter $H \in (0,1)$ and Hölder continuous drifts in $\mathcal{C}^\alpha$, for $\alpha \in [0,1]$. They obtained the strong error convergence rate $h^{(1/2+\alpha H)\wedge 1-\varepsilon}$, which holds whenever $\alpha \geq 0$ and $\alpha > 1 - 1/(2H)$. The latter condition is optimal in the sense that it corresponds to the existence and uniqueness result for (1.1) established in [7]. Our main contribution in this paper is an extension of their result to negative values of α.

First, we prove an extension of [1, Theorem 2.8] to dimension $d \geq 1$. Namely we prove that if b is in the Besov space $\mathcal{B}_p^\gamma$ and that $\gamma - d/p > 1/2 - 1/(2H)$, then there exists a weak solution (X, B) to (1.1) which has some Hölder regularity. This condition allows negative values of γ and therefore b can be a genuine distribution. Solutions to (1.1) are then understood as processes of the form $X_t = X_0 + K_t + B_t$, where K_t is the limit of $\int_0^t b^n(X_s)\, ds$, for any approximating sequence $(b^n)_{n\in\mathbb{N}}$. We see in particular that this approach is well suited for numerical approximation. Hence we propose a numerical scheme to approximate (1.1).

To that end, for a time-step h and a sequence $(b^n)_{n\in\mathbb{N}}$ that converges to b in a Besov sense, we consider the following Euler scheme that we define on the same probability space and with the same fBm B as X:

$$X_t^{h,n} = X_0 + \int_0^t b^n(X_{r_h}^{h,n})\, dr + B_t. \tag{1.2}$$

Choosing $b^n = g_{\frac{1}{n}} * b$ as a convolution of b with the Gaussian density, and for a careful choice of n as a function of h, we prove under the stronger condition $\gamma - d/p > 1 - 1/(2H)$ that the following rate of convergence holds

$$\sup_{t\in[0,1]} \|X_t - X_t^{h,n}\|_{L^m(\Omega)} \leq C h^{\frac{1}{2(1-\gamma+\frac{d}{p})}-\varepsilon}, \quad \forall h \in (0,1).$$

A more general version of this result is presented in Theorem 2.5 and discussed thereafter, in particular concerning the value of the rate. This extends the result of Butkovsky et al. [6] to $\alpha \equiv \gamma - d/p < 0$, and matches the $1/2 - \varepsilon$ rate of convergence obtained in the limit case $\gamma - d/p = 0$.

As a byproduct, we deduce that under the condition $\gamma - d/p > 1 - 1/(2H)$, X is in fact a strong solution and it is unique in a class of Hölder continuous processes.

Our proof relies on regularisation properties of the d-dimensional fBm and of the discrete-time fBm which somehow extend Davie's lemma [9, Prop. 2.1]. Namely, for functions f in Besov spaces of negative regularity (resp. bounded f for the discrete-time fBm), we obtain upper bounds on the moments of quantities such as $\int_s^t f(x+B_r)\,dr$ in terms of x and $(t-s)$, see Lemma 3.5, Propositions 5.1 and 5.4. These upper bounds are sharper than if B was replaced by any smooth function, which explains that we refer to regularisation properties of the fBm. The main tool to prove these results is the stochastic sewing lemma developed by Lê [20].

Organisation of the paper. We start with definitions and notations in Subsection 2.1, then state our main results in Subsection 2.2. In Section 3, we prove the existence of weak solutions to (1.1) and then use the convergence of the Euler scheme to establish strong existence and uniqueness. First we recall some Besov estimates in Subsection 3.1, then state a first regularisation property of the fBm in Subsection 3.2. We prove tightness and stability results in Subsection 3.3, and conclude the proof of weak existence in Subsection 3.4. The strong convergence of the numerical scheme (1.2) to the solution of (1.1) is established in Section 4. This proof relies strongly on the regularisation lemmas for fBm and discrete-time fBm which are stated and proven in Section 5. Finally in Section 6, we provide examples of SDEs that can be approximated with our result. We also run simulations of the scheme (1.2) to study its empirical rate of convergence and compare it with the theoretical result. In the Appendix, we gather some technical proofs based on the stochastic sewing lemma that are used to prove weak existence.

2 Framework and results

2.1 Notations and definitions

In this section, we define notations that are used throughout the paper. On a probability space $(\Omega,\mathcal{F},\mathbb{P})$, we denote by $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,1]}$ a filtration that satisfies the usual conditions. The conditional expectation given $\mathcal{F}_t$ is denoted by $\mathbb{E}^t$ when there is no risk of confusion on the underlying filtration. An $\mathbb{R}^d$-valued stochastic process $(X_t)_{t\in[0,1]}$ is said to be adapted if for all $t\in[0,1]$, X_t is $\mathcal{F}_t$-measurable. The $L^m(\Omega)$ norm of X_t is denoted by $\|X_t\|_{L^m}$.

For $\alpha\geq 0$, I a subset of $[0,1]$ and E a Banach space, we denote by $\mathcal{C}^\alpha_I E$ the space of E-valued mappings that are α-Hölder continuous on I. The corresponding semi-norm for a function $f:[0,1]\to E$ reads

$$[Z]_{\mathcal{C}^\alpha_I E} := \sup_{\substack{s,t\in I\\ t\neq s}} \frac{\|f_t-f_s\|_E}{|t-s|^\alpha}.$$

The space $L^m(\Omega)$, $m\in[1,\infty]$, will be simply denoted by L^m. In that case we denote by $\mathcal{C}^\alpha_I L^m$ the space of $L^m(\Omega)$-valued mappings that are α-Hölder continuous on I. For an $\mathbb{R}^d$-valued process X, the corresponding semi-norm is then denoted by $[Z]_{\mathcal{C}^\alpha_I L^m}$. When E is $\mathbb{R}$ or $\mathbb{R}^d$, we simply denote $\mathcal{C}^\alpha_I$ the corresponding space and when $\alpha=0$, we use the notation $\mathcal{C}_I$.

We write L^∞ for the space of bounded measurable functions and $L^\infty_I L^m := L^\infty(I, L^m(\Omega))$. The corresponding norm for a process $Z:[0,1]\times\Omega\to\mathbb{R}^d$ is

$$\|Z\|_{L^\infty_I L^m} := \sup_{s\in I}\|Z_s\|_{L^m}. \tag{2.1}$$

Moreover, for all $S,T\in[0,1]$, define the simplex $\Delta_{S,T}$ by

$$\Delta_{S,T}=\{(s,t)\in[S,T], s<t\}.$$

For a process $Z:\Delta_{0,1}\times\Omega\to\mathbb{R}^d$, we still write

$$[Z]_{\mathcal{C}^\alpha_I L^m} = \sup_{\substack{s,t\in I\\ s<t}} \frac{\|Z_{s,t}\|_{L^m}}{|t-s|^\alpha} \quad\text{and}\quad \|Z\|_{L^\infty_I L^m} = \sup_{s\in I}\|Z_{0,s}\|_{L^m}.$$

For a Borel-measurable function $f:\mathbb{R}^d\to\mathbb{R}^d$, denote the classical L^∞ and $\mathcal{C}^1$ norms of f by $\|f\|_\infty = \sup_{x\in\mathbb{R}^d}|f(x)|$ and $\|f\|_{\mathcal{C}^1} = \|f\|_\infty + \sup_{x\neq y}\frac{|f(x)-f(y)|}{d(x,y)}$.

Heat kernel. For any $t > 0$, denote by g_t the Gaussian kernel on $\mathbb{R}^d$ of variance t:

$$g_t(x) = \frac{1}{\sqrt{2\pi}\, t^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right),$$

and by G_t the associated Gaussian semigroup on $\mathbb{R}^d$: for $f : \mathbb{R}^d \to \mathbb{R}^d$,

$$G_t f(x) = \int_{\mathbb{R}^d} g_t(x-y)\, f(y)\, dy. \tag{2.2}$$

Besov spaces. We use the same definition of nonhomogeneous Besov spaces as in [5], which we write here for any dimension d. Let $\chi, \varphi : \mathbb{R}^d \to \mathbb{R}$ be the smooth radial functions which are given by [5, Proposition 2.10], with χ supported on a ball while φ is supported on an annulus. Let v_{-1} and v respectively be the inverse Fourier transform of χ and φ. Denote by $\mathcal{F}$ the Fourier transform and $\mathcal{F}^{-1}$ its inverse.

The nonhomogeneous dyadic blocks Δ_j, $j \in \mathbb{N} \cup \{-1\}$ are defined for any $\mathbb{R}^d$-valued tempered distribution u by

$$\Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u) \quad \text{and} \quad \Delta_j u = \mathcal{F}^{-1}\left(\varphi(2^{-j}\cdot)\mathcal{F} u\right) \quad \text{for } j \geq 0.$$

Let $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$. We denote by $\mathcal{B}_p^\gamma$ the nonhomogeneous Besov space $\mathcal{B}_{p,\infty}^\gamma(\mathbb{R}^d, \mathbb{R}^d)$ of $\mathbb{R}^d$-valued tempered distributions f such that

$$\|f\|_{\mathcal{B}_p^\gamma} = \sup_{j \geq -1} 2^{j\gamma} \|\Delta_j f\|_{L_p(\mathbb{R}^d)} < \infty.$$

Let $1 \leq p_1 \leq p_2 \leq \infty$. The space $\mathcal{B}_{p_1}^\gamma$ continuously embeds into $\mathcal{B}_{p_2}^{\gamma - d(1/p_1 - 1/p_2)}$, which we write as $\mathcal{B}_{p_1}^\gamma \hookrightarrow \mathcal{B}_{p_2}^{\gamma - d(1/p_1 - 1/p_2)}$, see e.g. [5, Prop. 2.71].

Finally, we denote by C a constant that can change from line to line and that does not depend on any parameter other than those specified in the associated lemma, proposition or theorem. When we want to make the dependence of C on some parameter a explicit, we will write $C(a)$.

To give a meaning to equation (1.1) with distributional drift, we first need to precise in which sense those drifts are approximated.

Definition 2.1. *Let $\gamma \in \mathbb{R}$, $p, m \in [1, \infty]$. We say that a sequence of smooth bounded functions $(b^n)_{n \in \mathbb{N}}$ converges to b in $\mathcal{B}_p^{\gamma-}$ as n goes to infinity if*

$$\begin{cases} \sup_{n \in \mathbb{N}} \|b^n\|_{\mathcal{B}_p^\gamma} \leq \|b\|_{B_p^\gamma} < \infty, \\ \lim_{n \to \infty} \|b^n - b\|_{\mathcal{B}_p^{\gamma'}} = 0, \quad \forall \gamma' < \gamma. \end{cases} \tag{2.3}$$

Following [26], in dimension $d = 1$, we define a notion of $\mathbb{F}$-fBm which extends the classical definition of $\mathbb{F}$-Brownian motion. There exists a one-to-one operator $\mathcal{A}_H$ (which can be written explicitly in terms of fractional derivatives and integrals, see [1, Definition 2.3]) such that for B an fBm, the process $W := \mathcal{A}_H B$ is a Brownian motion. Then we say that B is an $\mathbb{F}$-fBm if W is an $\mathbb{F}$-Brownian motion. In any dimension $d \geq 1$, we say that B is an $\mathbb{R}^d$-valued $\mathbb{F}$-fBm, if each component is an $\mathbb{F}$-fBm.

We are now ready to introduce the notions of solution to (1.1).

Definition 2.2. *Let $\gamma \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\gamma$, $T > 0$ and $X_0 \in \mathbb{R}^d$. As in [1], we define the following notions.*

- Weak solution: *a couple $((X_t)_{t \in [0,1]}, (B_t)_{t \in [0,1]})$ defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a weak solution to (1.1) on $[0, 1]$, with initial condition X_0, if*
  - *B is an $\mathbb{R}^d$-valued $\mathbb{F}$-fBm;*

– *X is adapted to $\mathbb{F}$;*
– *there exists an $\mathbb{R}^d$-valued process $(K_t)_{t\in[0,1]}$ such that, a.s.,*

$$X_t = X_0 + K_t + B_t \text{ for all } t \in [0,1]; \tag{2.4}$$

– *for every sequence $(b^n)_{n\in\mathbb{N}}$ of smooth bounded functions converging to b in $\mathcal{B}_p^{\gamma-}$, we have that*

$$\sup_{t\in[0,1]} \left| \int_0^t b^n(X_r)dr - K_t \right| \underset{n\to\infty}{\longrightarrow} 0 \text{ in probability.} \tag{2.5}$$

If the couple is clear from the context, we simply say that $(X_t)_{t\in[0,1]}$ is a weak solution.

- Pathwise uniqueness: *As in the classical literature on SDEs, we say that pathwise uniqueness holds if for any two solutions (X,B) and (Y,B) defined on the same filtered probability space with the same fBm B and same initial condition $X_0 \in \mathbb{R}^d$, X and Y are indistinguishable.*

- Strong solution: *A weak solution (X,B) such that X is $\mathbb{F}^B$-adapted is called a strong solution, where $\mathbb{F}^B$ denotes the filtration generated by B.*

2.2 Main results

Our first result is decomposed into two parts: first, it gives a condition for existence of a weak solution to (1.1) and therefore extends [1, Theorem 2.8] to the multidimensional setting. The proof is presented in Section 3. The second part gives existence and uniqueness of a strong solution under stronger assumptions, and will be a consequence of the convergence of the Euler scheme in Theorem 2.5. Thus it provides a multidimensional extension of [1, Theorem 2.9] through a completely different proof. However the critical case $\gamma - d/p = 1 - 1/(2H)$ is not addressed here.

Theorem 2.3. *Let $\gamma \in \mathbb{R}$, $p \in [1,\infty]$ and $b \in \mathcal{B}_p^\gamma$.*

(a) Assume that

$$\gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}. \tag{2.6}$$

Then there exists a weak solution X to (1.1) such that X_0 is $\mathcal{F}_0$-measurable and $X - B \in \mathcal{C}^\kappa_{[0,1]} L^m$ for any $\kappa \in (0, 1 + H(\gamma - \frac{d}{p}) \wedge 0] \setminus \{1\}$ and $m \geq 2$.

(b) Assume that

$$H < \frac{1}{2} \quad \text{and} \quad 1 - \frac{1}{2H} < \gamma - \frac{d}{p} < 0. \tag{2.7}$$

Then there exists a strong solution X to (1.1) such that $X - B \in \mathcal{C}^{1/2+H}_{[0,1]} L^m$ for any $m \geq 2$. Besides, pathwise uniqueness holds in the class of all solutions X such that $X - B \in \mathcal{C}^{1/2+H}_{[0,1]} L^2$.

The proof of Theorem 2.3 is given in Section 3.

Remark 2.4. *This theorem is consistent with [7], where it is proven that for b in the Hölder space $\mathcal{C}^\alpha$ ($= \mathcal{B}^\alpha_\infty$ when $\alpha \notin \mathbb{N}$), path-by-path existence and uniqueness hold for $\alpha > 1 - 1/(2H)$. The solution in [7] is in the sense of nonlinear Young differential equations, which can be compared to the present definition (see [1, Theorem 2.14]).*

Let $(b^n)_{n\in\mathbb{N}}$ be a sequence of smooth functions that converges to b in $\mathcal{B}_p^{\gamma-}$. Consider the Euler scheme (1.2) associated to (1.1) with a time-step $h \in (0,1)$. The main result of this paper is the following theorem.

Theorem 2.5. *Let $H < 1/2$, $\gamma \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\gamma$ and assume (2.7). For $m \in [2, \infty]$, let (X, B) be a weak solution to (1.1) such that $X - B \in \mathcal{C}^{1/2+H}_{[0,1]} L^m$ and $X_0 \in L^m(\Omega)$ is $\mathcal{F}_0$-measurable. Let $\varepsilon \in (0, 1/2)$. Then there exists a constant C that depends only on $m, p, \gamma, \varepsilon, \|b\|_{\mathcal{B}_p^\gamma}$ such that for all $h \in (0, 1)$ and $n \in \mathbb{N}$, the following bound holds:*

$$[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C \left(\|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} + \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_\infty \|b^n\|_{\mathcal{C}^1} h^{1-\varepsilon} \right). \tag{2.8}$$

Obvisouly, the previous inequality implies that

$$\sup_{t \in [0,1]} \|X_t - X_t^{h,n}\|_{L^m} \leq C \left(\|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} + \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_\infty \|b^n\|_{\mathcal{C}^1} h^{1-\varepsilon} \right). \tag{2.9}$$

Although the Hurst parameter does not appear in the upper bound of the previous inequality, the first term $\|b^n - b\|_{\mathcal{B}_p^{\gamma-1}}$ does depend implicitly on H through the condition (2.7). Observe also that the second term, $\|b^n\|_\infty h^{1/2-\varepsilon}$, corresponds to the optimal rate of convergence found in [6].

Since b^n converges to b in $\mathcal{B}_p^{\gamma-}$ for $\gamma < 0$, we expect in general $\|b^n\|_\infty$ and $\|b^n\|_{\mathcal{C}^1}$ to diverge as $n \to \infty$. Therefore, it is important to choose carefully the sequence $(b^n)_{n \in \mathbb{N}}$ to obtain a good rate of convergence of the numerical scheme.

Choosing $b^n = G_{\frac{1}{n}}$ for $n \in \mathbb{N}^*$, we have thanks to Lemma 3.2 that for $\gamma - d/p < 0$,

$$\|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} \leq C \, \|b\|_{\mathcal{B}_p^\gamma} \; n^{-\frac{1}{2}}, \tag{2.10}$$

$$\|b^n\|_\infty \leq C \, \|b\|_{\mathcal{B}_p^\gamma} \; n^{-\frac{1}{2}(\gamma - \frac{d}{p})}, \tag{2.11}$$

$$\|b^n\|_{\mathcal{C}^1} \leq C \, \|b\|_{\mathcal{B}_p^\gamma} \, n^{\frac{1}{2}} \; n^{-\frac{1}{2}(\gamma - \frac{d}{p})}. \tag{2.12}$$

Using these results in (2.8) and optimising over m and h, we deduce the following corollary.

Corollary 2.6. *Let the same assumptions and notations as in Theorem 2.5 hold. Let $h \in (0, 1/2)$ and define*

$$n_h = \lfloor h^{-\frac{1}{1-\gamma+\frac{d}{p}}} \rfloor \quad \text{and} \quad b^{n_h} = G_{\frac{1}{n_h}} b.$$

Then there exists a constant C that depends only on $m, p, \gamma, \varepsilon, \|b\|_{\mathcal{B}_p^\gamma}$ such that the following bounds hold:

$$[X - X^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C \, h^{\frac{1}{2(1-\gamma+\frac{d}{p})} - \varepsilon}, \tag{2.13}$$

$$\sup_{h \in (0,1)} [X^{h,n_h} - B]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} < \infty. \tag{2.14}$$

For instance, if each component of b is a signed measure, then $b \in \mathcal{B}_1^0$ (see [5, Proposition 2.39]). Hence the previous result yields a rate $\frac{1}{2(1+d)} - \varepsilon$ which holds for $H < \frac{1}{2(1+d)}$.

Remark 2.7. *In Theorem 2.5, we construct the Euler scheme on a particular probability space that is given in an abstract way by Theorem 2.3. From Corollary 2.6, we deduce (see Subsection 3.5) that X is in fact a strong solution. It is then possible to construct the Euler scheme on any probability space (rich enough to contain an $\mathbb{F}$-fBm). This is important for practical simulations.*

For $\gamma - d/p > 0$, $\mathcal{B}_p^\gamma$ is continuously embedded in the Hölder space $\mathcal{C}^{\gamma - d/p}$. In [6], it was proved that the Euler scheme achieves a rate $1/2 + H(\gamma - d/p) - \varepsilon$. Moreover, if b a bounded measurable function, the rate is $1/2 - \varepsilon$. To close the gap between the present results and [6], we handle the case $\gamma - d/p = 0$. Recall that $\mathcal{B}_p^{d/p}$ is continuously embedded into $\mathcal{B}_\infty^0$, so it is equivalent to work with $\gamma = 0$ and $p = +\infty$. Note that $\mathcal{B}_\infty^0$ contains strictly $L^\infty(\mathbb{R}^d)$ (see e.g. [5, p.99]) which was the space considered in [6].

Let $b \in \mathcal{B}_\infty^0$. By the definition of Besov spaces, we know that $b \in \mathcal{B}_\infty^{-\eta}$ for all $\eta > 0$. Choosing η small enough so that $-\eta > 1 - 1/(2H)$, we can apply Theorem 2.3 and Theorem 2.5, and obtain a rate of convergence as in Corollary 2.6 when $b^n = G_{\frac{1}{n}} b$. This is summarized in the following Corollary.

Corollary 2.8. *Let the same assumptions of Theorem 2.5 hold. Let B be an $\mathbb{F}$-fBm with $H < 1/2$, $b \in \mathcal{B}^0_\infty$ and $m \geq 2$. There exists a strong solution X to (1.1) such that $X - B \in \mathcal{C}^{1/2+H}_{[0,1]} L^m$. Besides, pathwise uniqueness holds in the class of solutions X such that $X - B \in \mathcal{C}^{1/2+H}_{[0,1]} L^2$. Let $\varepsilon \in (0, 1/2)$ and $h \in (0, 1/2)$. There exists a constant C that depends only on $m, \varepsilon, \|b\|_{B^0_\infty}$ such that the following bound holds:*

$$[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C \left(\|b^n - b\|_{\mathcal{B}^{-1}_\infty} + \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_\infty \|b^n\|_{\mathcal{C}^1} h^{1-\varepsilon} \right).$$

Moreover, for $n_h = \lfloor h^{-1} \rfloor$ and $b^{n_h} = G_{\frac{1}{n_h}} b$, we have

$$[X - X^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C h^{\frac{1}{2}-\varepsilon},$$

$$\sup_{h \in (0,1)} [X^{h,n_h} - B]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} < \infty.$$

Theorem 2.5, Corollary 2.6 and Corollary 2.8 are proven in Section 4.

2.3 Discussion on the approach and results

The main novelty of the paper is that we can treat SDEs with unbounded or even distributional drifts. Usually for bounded drifts, one can use Girsanov's theorem (see e.g. [6, Lemma 4.2]), which leads to an exponential dependence on $\|b^n\|_\infty$. Using this approach would have led at best to a logarithmic rate of convergence. Instead, we use the stochastic sewing lemma to regularise integrals of functions of the discrete noise $\{B_{t_h}\}_{t \geq 0}$, see Proposition 5.4. The price to pay is the $\mathcal{C}^1$ norm of b^n, which can be compensated by h.

Let us make a few comments on the rate of convergence obtained in Corollaries 2.6 and 2.8:

- For a fixed γ, p and d, one can chose H close to $\frac{1}{2(1-\gamma+\frac{d}{p})}$ from below, and get an order of convergence that will be $\frac{1}{2(1-\gamma+\frac{d}{p})} - \varepsilon \approx H - \varepsilon$.
- For a fixed H, one can take $b \in \mathcal{B}^{1-\frac{1}{2H}+\varepsilon}_\infty$ for any $\varepsilon > 0$, and get an order of convergence that will be close to H.
- The order of convergence is $\frac{1}{2} - \varepsilon$ when $\gamma - \frac{d}{p} = 0$, for any $H < \frac{1}{2}$.

In view of [6], one could imagine that the order of convergence for $\gamma - d/p \leq 0$ would still be $1/2 + H(\gamma - d/p)$. In particular, we point out that our assumption $\gamma - d/p > -1/(2H) + 1$ is equivalent to

$$\frac{1}{2} + H\left(\gamma - \frac{d}{p}\right) > \frac{1}{2(1-\gamma+\frac{d}{p})}.$$

Finally, the orders of convergence obtained here and in [6] are summarized in Table 1.

The drift	$\gamma - \frac{d}{p} \to (1 - \frac{1}{2H})^+$	$\gamma - \frac{d}{p} \in (1 - \frac{1}{2H}, 0)$	$\gamma - \frac{d}{p} = 0$	$\gamma - \frac{d}{p} > 0$
Convergence	$H - \varepsilon$	$\frac{1}{2(1-\gamma+\frac{d}{p})} - \varepsilon$	$\frac{1}{2} - \varepsilon$	$\left(\frac{1}{2} + H\left(\gamma - \frac{d}{p}\right)\right) \wedge 1 - \varepsilon$

Table 1: Order of convergence of the Euler scheme to the solution of the SDE (1.1) for $\gamma - d/p > 1 - 1/(2H)$.

3 Existence and uniqueness of solutions

In this section, we prove Theorem 2.3. The proof of Theorem 2.3(a) follows the same lines as the proof of Theorem 2.8 of [1], but requires extensions of some technical lemmas concerning the regularising effects of the d-dimensional fractional Brownian motion. Subsection 3.5 is dedicated to the proof of Theorem 2.3(b).

3.1 Besov estimates

The first of these extensions concern estimates of shift of distributions in Besov spaces. It is an extension to $\mathbb{R}^d$ of Lemma A.2 in [3] and follows its proof exactly, so we omit it.

Lemma 3.1. *Let f be a tempered distribution on $\mathbb{R}^d$ and let $\gamma \in \mathbb{R}$, $p \in [1, \infty]$. Then for any $a_1, a_2, a_3 \in \mathbb{R}^d$ and $\alpha, \alpha_1, \alpha_2 \in [0, 1]$, one has*

(i) $\|f(a + \cdot)\|_{\mathcal{B}_p^\gamma} \leq \|f\|_{\mathcal{B}_p^\gamma}$.

(ii) $\|f(a_1 + \cdot) - f(a_2 + \cdot)\|_{\mathcal{B}_p^\gamma} \leq C|a_1 - a_2|^\alpha \|f\|_{\mathcal{B}_p^{\gamma+\alpha}}$.

(iii) $\|f(a_1 + \cdot) - f(a_2 + \cdot) - f(a_3 + \cdot) + f(a_3 + a_2 - a_1 + \cdot)\|_{\mathcal{B}_p^\gamma} \leq C|a_1 - a_2|^{\alpha_1} |a_1 - a_3|^{\alpha_2} \|f\|_{\mathcal{B}_p^{\gamma+\alpha_1+\alpha_2}}$.

Then we have the following estimates for the Gaussian semigroup in Besov spaces. They are either borrowed or adapted from [3, 5].

Lemma 3.2. *Let $\gamma \in \mathbb{R}$, $p \in [1, \infty]$ and $f \in \mathcal{B}_p^\gamma$. Then*

(i) If $\gamma < 0$, $\|G_t f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{\mathcal{B}_p^\gamma} t^{\frac{\gamma}{2}}$, for all $t > 0$.

(ii) If $\gamma - \frac{d}{p} < 0$, $\|G_t f\|_\infty \leq C \|f\|_{\mathcal{B}_p^\gamma} t^{\frac{1}{2}(\gamma - \frac{d}{p})}$, for all $t > 0$.

(iii) $\|G_t f - f\|_{\mathcal{B}_p^{\gamma-\varepsilon}} \leq C t^{\frac{\varepsilon}{2}} \|f\|_{\mathcal{B}_p^\gamma}$ for all $\varepsilon \in (0, 1]$ and $t > 0$. In particular, it follows that $\lim_{t\to 0} \|G_t f - f\|_{\mathcal{B}_p^{\tilde{\gamma}}} = 0$ for every $\tilde{\gamma} < \gamma$.

(iv) $\sup_{t>0} \|G_t f\|_{\mathcal{B}_p^\gamma} \leq \|f\|_{\mathcal{B}_p^\gamma}$.

(v) If $\gamma - \frac{d}{p} < 0$, $\|G_t f\|_{\mathcal{C}^1} \leq C \|f\|_{\mathcal{B}_p^\gamma} t^{\frac{1}{2}(\gamma - \frac{d}{p} - 1)}$ for all $t > 0$.

Proof. (i) The proof of Lemma A.3(i) in [3] extends right away to dimension $d \geq 1$.

(ii) Using (i) for $\gamma - \frac{d}{p}$ instead of γ and the embedding $\mathcal{B}_p^\gamma \hookrightarrow \mathcal{B}_\infty^{\gamma - \frac{d}{p}}$, there is

$$\|G_t f\|_{L^\infty(\mathbb{R}^d)} \leq C \|f\|_{\mathcal{B}_\infty^{\gamma-d/p}} t^{\frac{1}{2}(\gamma - \frac{d}{p})} \leq C \|f\|_{\mathcal{B}_p^\gamma} t^{\frac{1}{2}(\gamma - \frac{d}{p})}.$$

(iii) This is a direct adaptation of Lemma A.3(ii) in [3] to dimension $d \geq 1$ that we detail briefly. From [24, Lemma 4], we have that for g such that the support of $\mathcal{F}g$ is in a ball of radius $\lambda \geq 1$, there is for all $t \geq 0$,

$$\|G_t g - g\|_{L^p(\mathbb{R}^d)} \leq C (t\lambda^2 \wedge 1) \|g\|_{L^p(\mathbb{R}^d)}.$$

For any $j \geq -1$, the support of $\mathcal{F}(\Delta_j f)$ is included in a ball of radius 2^j. Hence,

$$\begin{aligned} 2^{j(\gamma-\varepsilon)} \|G_t(\Delta_j f) - \Delta_j f\|_{L^p(\mathbb{R}^d)} &\leq C\, 2^{j(\gamma-\varepsilon)} (t 2^{2j} \wedge 1) \|\Delta_j f\|_{L^p(\mathbb{R}^d)} \\ &\leq C\, 2^{-j\varepsilon} (t 2^{2j} \wedge 1)^{\frac{\varepsilon}{2}}\, 2^{j\gamma} \|\Delta_j f\|_{L^p(\mathbb{R}^d)} \\ &\leq C\, t^{\frac{\varepsilon}{2}}\, 2^{j\gamma} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

The result follows.

(iv) – (v) The proof is the same as in the one-dimensional, see [3, Lemma A.3(iii)] and [3, Lemma A.3(iv)].

□

The next lemma describes some time regularity estimates of random functions of the fractional Brownian motion in Besov norms.

Lemma 3.3. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and B be an $\mathbb{F}$-fBm. Let $\gamma < 0$, $p \in [1, \infty]$ and $e \in \mathbb{N}^*$. Let $f : \mathbb{R}^d \times \mathbb{R}^e \to \mathbb{R}^d$ be a bounded measurable function. Then there exists a constant $C > 0$ such that for any $(s, t) \in \Delta_{0,1}$ and for Ξ an $\mathcal{F}_s$-measurable $\mathbb{R}^e$-valued random variable satisfying $\|f(\cdot, \Xi)\|_{\mathcal{C}^1} < \infty$ almost surely, there is*

(i) $\mathbb{E}^s[f(B_t, \Xi)] = G_{\sigma^2_{s,t}} f(\mathbb{E}^s[B_t], \Xi)$, *where G is the Gaussian semigroup introduced in* (2.2) *and* $\sigma^2_{s,t} := \mathrm{Var}\,(B^{(i)}_t - \mathbb{E}^s[B^{(i)}_t])$, *for any component $B^{(i)}$ of the fBm;*

(ii) $|\mathbb{E}^s f(B_t, \Xi)| \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}^\gamma_p} (t-s)^{H(\gamma - \frac{d}{p})}$;

(iii) $\|f(B_t, \Xi) - \mathbb{E}^s f(B_t, \Xi)\|_{L^1} \leq C \| \|f(\cdot, \Xi)\|_{\mathcal{C}^1} \|_{L^2} (t-s)^H$.

Proof. The proofs of (i), (ii) and (iii) are similar to (a), (b) and (c) in [1, Lemma 5.1] and we do not reproduce the proofs. We just note that the local nondeterminism property is used on each independent component of B, and that we now rely on Lemma 3.1(i). □

3.2 Regularisation effect of the d-dimensional fBm

Let $0 \leq S \leq T \leq 1$. Let $A : \Delta_{S,T} \to L^m$. For every triplet of times (s, u, t) such that $S \leq s \leq u \leq t \leq T$, we denote

$$\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}.$$

We use the stochastic sewing lemma of [20] (recalled in Lemma A.1) to establish the key regularisation result (Proposition 3.5) that will be used to prove existence of weak solutions. Note that the results in this subsection and the next one have slightly simpler proofs than in the one-dimensional framework developed in [1]. This is because we are not aiming here at proving uniqueness, which requires milder assumptions on the coefficients γ, p and H than here and requires to work with the semi-norm $[\cdot]_{\mathcal{C}^\tau_{[s,t]} L^{m,n}}$ instead of $[\cdot]_{\mathcal{C}^\tau_{[s,t]} L^m}$ that is used here.

First, we have the following lemma, which extends [1, Lemma D.2] to dimension $d \geq 1$. Its proof, which is also close to the proof of [1, Lemma D.2], is postponed to the Appendix A.1.

Lemma 3.4. *Let $\gamma - d/p \in (-1/(2H), 0)$, $m \geq 2$ and $p \geq 1$. Then there exists a constant $C > 0$ such that for any $0 \leq S \leq T$, any $\mathcal{F}_S$-measurable random variable Ξ in $\mathbb{R}^e$ and any bounded measurable function $f : \mathbb{R}^d \times \mathbb{R}^e \to \mathbb{R}^d$ fulfilling*

(i) $\mathbb{E}\left[\|f(\cdot, \Xi)\|^2_{\mathcal{C}^1}\right] < \infty$;

(ii) $\mathbb{E}\left[\|f(\cdot, \Xi)\|^m_{\mathcal{B}^\gamma_p}\right] < \infty$,

we have for any $t \in [S, T]$ that

$$\Big\| \int_S^t f(B_r, \Xi)\, dr \Big\|_{L^m} \leq C\, \| \|f(\cdot, \Xi)\|_{\mathcal{B}^\gamma_p} \|_{L^m}\, (t-S)^{1+H(\gamma - \frac{d}{p})}. \tag{3.1}$$

As a consequence of Lemmas A.1 and 3.4, we get the following property of regularisation of the d-dimensional fBm. It can be compared to [1, Lemma 7.1], which is stated for one-dimensional processes but involves a slightly more accurate quantity. The proof is postponed to Appendix A.2.

Proposition 3.5. *Let $(\psi_t)_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued stochastic process adapted to $\mathbb{F}$. Let $m \in [2, \infty)$, $p \in [1, \infty]$ and $\gamma < 0$ such that $(\gamma - d/p)H > -1/2$. Let $\tau \in (0, 1)$ such that $H(\gamma - d/p - 1) + \tau > 0$. There exists a constant $C > 0$ such that for any $f \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\gamma$ and any $(s, t) \in \Delta_{0,1}$ we have*

$$\begin{aligned}\Big\| \int_s^t f(B_r + \psi_r)\, dr \Big\|_{L^m} \leq & C \, \|f\|_{\mathcal{B}_p^\gamma} (t-s)^{1+H(\gamma - \frac{d}{p})} \\ & + C \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{\mathcal{C}^\tau[s,t] L^m} \, (t-s)^{1+H(\gamma-\frac{d}{p}-1)+\tau}.\end{aligned} \tag{3.2}$$

3.3 Tightness and stability

The proof of existence of a weak solution is based on a classical argument: first, we construct a tight approximating sequence of processes (Proposition 3.8), then we prove the stability, i.e. that any converging subsequence is a solution of the SDE (1.1) (Proposition 3.9).

First we need the two following *a priori* estimates, which are direct consequences of Proposition 3.5: Lemma 3.6 (resp. Lemma 3.7) extends [1, Lemma 7.3] (resp. [1, Lemma 7.4]) to dimension $d \geq 1$.

Lemma 3.6. *Let $m \in [2, \infty)$ and assume that $1/2 - 1/(2H) < \gamma - d/p < 0$. There exists $C > 0$, such that, for any $b \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\gamma$,*

$$[X - B]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[0,1]} L^m} \leq C \, (1 + \|b\|^2_{\mathcal{B}_p^\gamma}), \tag{3.3}$$

where X is the strong solution to (1.1) *with drift b.*

Proof. Without loss of generality, assume that $X_0 = 0$ and denote $K = X - B$. Then $[K]_{\mathcal{C}^\tau_{[0,1]} L^m}$ is finite for any $\tau \in (0, 1]$ as $|K_t - K_s| = |\int_s^t b(B_r + K_r)\, dr| \leq \|b\|_\infty |t - s|$.

We aim to apply Proposition 3.5 with $\tau = 1 + H(\gamma - d/p)$ and considering $b \in \mathcal{B}_\infty^{\gamma - d/p}$ after an embedding (so that $\gamma - d/p < 0$). Remark that $\tau - H > 1/2 - H/2 > 0$, thus $\tau \in (0, 1)$. In addition, $H(\gamma - d/p) > H/2 - 1/2 > -1/2$ and $H(\gamma - d/p - 1) + \tau > 0$. So, the assumptions of Proposition 3.5 are fulfilled. Then we get

$$\begin{aligned}\|K_t - K_s\|_{L^m} &\leq C \, \|b\|_{\mathcal{B}_\infty^{\gamma-d/p}} \Big((t-s)^{1+H(\gamma-\frac{d}{p})} + [K]_{\mathcal{C}^\tau_{[s,t]} L^m} (t-s)^{1+H(\gamma-\frac{d}{p})+\tau-H} \Big) \\ &= C \|b\|_{\mathcal{B}_p^\gamma} (t-s)^{1+H(\gamma-\frac{d}{p})} \Big(1 + [K]_{\mathcal{C}^\tau_{[s,t]} L^m} (t-s)^{\tau - H} \Big).\end{aligned} \tag{3.4}$$

Choose $l = (4C\|b\|_{\mathcal{B}_p^\gamma})^{1/(H-\tau)}$ so that $C\|b\|_{\mathcal{B}_p^\gamma} l^{\tau-H} < 1/2$. Let $u \in [0, 1]$. Divide both sides in (3.4) by $(t - s)^{1+H(\gamma-d/p)}$ and take the supremum over $(s, t) \in \Delta_{u,(u+l)\wedge 1}$ to get

$$[K]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[u,(u+l)\wedge 1]} L^m} \leq \Big(C\|b\|_{\mathcal{B}_p^\gamma} + \frac{1}{2} [K]_{\mathcal{C}^\tau_{[u,(u+l)\wedge 1]} L^m} \Big),$$

and therefore

$$[K]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[u,(u+l)\wedge 1]} L^m} \leq 2C\|b\|_{\mathcal{B}_p^\gamma}.$$

The end of the proof consists in iterating the previous inequality in order to control the Hölder norm on the whole interval $[0, 1]$, and is completely identical to the proof of [1, Lemma 7.3]. $\square$

Lemma 3.7. *Let $b, h \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\gamma$. Let X be the strong solution to* (1.1) *with drift b. Let $\delta \in (0, 1 + H(\gamma - d/p))$. Then there exists a constant $C > 0$ which is independent of X_0, b and h, and a nonnegative random variable Z which satisfies $\mathbb{E}[Z] \leq C\|h\|_{\mathcal{B}_p^\gamma} (1 + \|b\|^2_{\mathcal{B}_p^\gamma})$ such that*

$$\Big| \int_s^t h(X_r)\, dr \Big| \leq Z \, |t - s|^\delta. \tag{3.5}$$

The proof relies on Proposition 3.5, Lemma 3.6 and Kolmogorov's continuity criterion. We omit it, as it is identical to the proof of [1, Lemma 7.4].

We now obtain tightness of the sequence that approximates X.

Proposition 3.8. *Let $(b^n)_{n\in\mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\gamma-}$. For $n\in\mathbb{N}$, let X^n be the strong solution to (1.1) with initial condition X_0 and drift b^n. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $(X^{n_k},B)_{k\in\mathbb{N}}$ converges weakly in the space $[\mathcal{C}_{[0,1]}(\mathbb{R}^d)]^2$.*

Proof. This short proof is close to the proof of [1, Proposition 7.5]. We reproduce it here for the reader's convenience.

Let $K_t^n := \int_0^t b^n(X_r^n)\,dr$. For $M>0$ and some $\delta\in(0,1+H(\gamma-d/p))$, let

$$A_M := \{f\in\mathcal{C}_{[0,1]} : f(0)=0,\ |f(t)-f(s)|\leq M(t-s)^\delta,\ \forall (s,t)\in\Delta_{0,1}\}.$$

By Arzelà-Ascoli's theorem, A_M is compact in $\mathcal{C}_{[0,1]}$. Applying Lemma 3.7 to $h=b^n$, this gives a nonnegative random variable Z^n such that $\mathbb{E}[Z^n]\leq C\|b^n\|_{\mathcal{B}_p^\gamma}(1+\|b^n\|^2_{\mathcal{B}_p^\gamma})$ and (3.5) is satisfied. Thus by Markov's inequality we get

$$\begin{aligned}\mathbb{P}(K^n\notin A_M) &\leq \mathbb{P}(\exists (s,t)\in\Delta_{0,1} : |K^n_{s,t}|> M(t-s)^\delta)\\ &\leq \mathbb{P}(Z^n>M)\\ &\leq C\sup_{n\in\mathbb{N}}\|b^n\|_{\mathcal{B}_p^\gamma}\,(1+\sup_{n\in\mathbb{N}}\|b^n\|^2_{\mathcal{B}_p^\gamma})\,M^{-1}.\end{aligned}$$

Hence, the sequence $(K^n)_{n\in\mathbb{N}}$ is tight in $\mathcal{C}_{[0,1]}$. So $(K^n,B)_{n\in\mathbb{N}}$ is tight in $(\mathcal{C}_{[0,1]})^2$. Thus by Prokhorov's Theorem, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $(K^{n_k},B)_{k\in\mathbb{N}}$ converges weakly in the space $(\mathcal{C}_{[0,1]})^2$, and so does $(X^{n_k},B)_{k\in\mathbb{N}}$. $\square$

Finally, the stability is expressed in the following proposition, which extends [1, Proposition 7.7] to dimension $d\geq 1$.

Proposition 3.9. *Let $(\tilde{b}^k)_{k\in\mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\gamma-}$. Let $\tilde{B}^k$ have the same law as B. We consider $\tilde{X}^k$ the strong solution to (1.1) for $B=\tilde{B}^k$, initial condition X_0 and drift $\tilde{b}^k$. Assume that there exist stochastic processes $\tilde{X},\tilde{B}:[0,1]\to\mathbb{R}^d$ such that $(\tilde{X}^k,\tilde{B}^k)_{k\in\mathbb{N}}$ converges to $(\tilde{X},\tilde{B})$ on $[\mathcal{C}_{[0,1]}(\mathbb{R}^d)]^2$ in probability. Then $\tilde{X}$ fulfills (2.4) and (2.5) from Definition 2.2 and for any $m\in[2,\infty)$, there exists $C>0$ such that*

$$[\tilde{X}-\tilde{B}]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[0,1]}(L^m)}\leq C\,(1+\sup_{k\in\mathbb{N}}\|\tilde{b}^k\|^2_{\mathcal{B}_p^\gamma})<\infty. \tag{3.6}$$

The proof is postponed to Appendix A.3.

3.4 Proof of Theorem 2.3 *(a)*

Let $(b^n)_{n\in\mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\gamma-}$. By Proposition 3.8, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $(X^{n_k},B)_{k\in\mathbb{N}}$ converges weakly in $(\mathcal{C}_{[0,1]}(\mathbb{R}^d))^2$. Without loss of generality, we assume that $(X^n,B)_{n\in\mathbb{N}}$ converges weakly. By the Skorokhod representation Theorem, there exists a sequence of random variables $(Y^n,\hat{B}^n)_{n\in\mathbb{N}}$ defined on a common probability space $(\hat{\Omega},\hat{\mathcal{F}},\hat{P})$, such that

$$\mathrm{Law}(Y^n,\hat{B}^n)=\mathrm{Law}(X^n,B),\ \forall n\in\mathbb{N}, \tag{3.7}$$

and $(Y^n,\hat{B}^n)$ converges a.s. to some $(Y,\hat{B})$ in $(\mathcal{C}_{[0,1]}(\mathbb{R}^d))^2$. As X^n solves (1.1) with drift b^n, we know by (3.7) that Y^n also solves (1.1) with drift b^n and $\hat{B}^n$ instead of B. As X^n is a strong solution, we have that X^n is adapted to $\mathbb{F}^B$. Hence by (3.7), we know that Y^n is adapted to $\mathbb{F}^{\hat{B}^n}$ as the conditional laws of Y^n and X^n agree and therefore it is a strong solution to (1.1) with $\hat{B}^n$ instead of B.

By Proposition 3.9, we know that Y fulfills (2.4) and (2.5) from Definition 2.2 with $\hat{B}$ instead of B and it is adapted with respect to the filtration $\hat{\mathbb{F}}$ defined by $\hat{\mathcal{F}}_t := \sigma(Y_s,\hat{B}_s, s\in[0,t])$. It remains to check that $\hat{B}$ is an $\hat{\mathbb{F}}$-fBm, which is completely analogous to the one-dimensional case treated in the proof of Theorem 2.8 in [1]. Hence Y is a weak solution.

Finally, (3.6) gives that

$$[Y-\hat{B}]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[0,1]}(L^m)}<\infty,$$

which concludes the proof.

3.5 Proof of Theorem 2.3 *(b)*

Although it will be proven in the next sections, we will use Theorem 2.5 to prove Theorem 2.3 *(b)*.

Assuming the strong conditions (2.7) on H, γ and p, we let (X,B) and a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a weak solution to (1.1) given by Theorem 2.3 *(a)*. On this probability space and with the same fBm B, we define the Euler scheme $(X^{h,n})_{h>0,n\in\mathbb{N}}$.

First, observe that the Euler scheme $X^{h,n}$ is $\mathbb{F}^B$-adapted. In view of (2.9), $X^{h,n}_t$ converges to X_t in L^m, for each $t\in[0,1]$. Hence X_t is $\mathcal{F}^B_t$-measurable and X is therefore a strong solution.

As for the uniqueness, if X and Y are two processes such that $X-B\in\mathcal{C}^{1/2+H}_{[0,1]}L^2$ and $Y-B\in\mathcal{C}^{1/2+H}_{[0,1]}L^2$, then by Theorem 2.5, $X^{h,n}$ approximates both X and Y. So X and Y are modifications of one another. Since they are continuous processes, they are indistinguishable.

4 Convergence of the Euler scheme

4.1 Proof of Theorem 2.5

In this section, we prove Theorem 2.5. Let (X,B) be a weak solution to (1.1) defined on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. On this probability space and with the same fBm B, we define the Euler scheme $(X^{h,n})_{h>0,n\in\mathbb{N}}$.

For all $t>0$, recall from (2.4) that $K_t:=X_t-B_t-X_0$ and define

$$K^n_t:=\int_0^t b^n(X_r)\,dr \text{ and } K^{h,n}_t:=\int_0^t b^n(X^{h,n}_{r_h})\,dr. \tag{4.1}$$

Assume that the solution X starts from an $\mathcal{F}_0$-measurable random variable ξ and that $X^{h,n}$ starts from another $\mathcal{F}_0$-measurable random variable η. Allowing different random initial conditions will be necessary later to deduce the result of the theorem by reasoning on short time intervals. With these notations in mind, we set the notation for the error as

$$\mathcal{E}^{h,n}_t:=X_t-X^{h,n}_t,\quad t\geq 0.$$

Decompose the error as

$$\begin{aligned}
[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac12}_{[s,t]}L^m} &\leq [K-K^n]_{\mathcal{C}^{\frac12}_{[s,t]}L^m}+[K^n-K^{h,n}]_{\mathcal{C}^{\frac12}_{[s,t]}L^m}\\
&=[K-K^n]_{\mathcal{C}^{\frac12}_{[s,t]}L^m}+[E^{h,n}]_{\mathcal{C}^{\frac12}_{[s,t]}L^m}\\
&\leq [K-K^n]_{\mathcal{C}^{\frac12}_{[s,t]}L^m}+[E^{1,h,n}]_{\mathcal{C}^{\frac12}_{[s,t]}L^m}+[E^{2,h,n}]_{\mathcal{C}^{\frac12}_{[s,t]}L^m},
\end{aligned} \tag{4.2}$$

where for all $s<t$, we denote

$$\begin{aligned}
E^{h,n}_{s,t}&:=K^n_t-K^n_s-(K^{h,n}_t-K^{h,n}_s),\\
E^{1,h,n}_{s,t}&:=\int_s^t (b^n(\xi+K_r+B_r)-b^n(\eta+K^{h,n}_r+B_r))\,dr,\\
E^{2,h,n}_{s,t}&:=\int_s^t (b^n(\eta+K^{h,n}_r+B_r)-b^n(\eta+K^{h,n}_{r_h}+B_{r_h}))\,dr.
\end{aligned}$$

In order to prove Theorem 2.5, we will provide bounds on the quantities that appear in (4.2), and combine them eventually. The two technical bounds that concern $E^{1,h,n}$ and $E^{1,h,n}$ will be stated in Corollaries 5.2 and 5.6, and proven in Section 5.

Bound on $K - K^n$. By Proposition 3.5 (taking $f = b^k - b^n$), we have that for all $n, k \in \mathbb{N}$, $s < t$, and $(u, v) \in \Delta_{s,t}$,

$$\|K_v^k - K_u^k - K_v^n + K_u^n\|_{L^m} \leq C\|b^k - b^n\|_{\mathcal{B}_p^{\gamma-1}}(v-u)^{1+H(\gamma-1-\frac{d}{p})}.$$

Hence $(K_v^k - K_u^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^m(\Omega)$ and therefore it converges. We also know by definition of X that $K_v^k - K_u^k$ converges in probability to $K_v - K_u$. Thus $K_v^k - K_u^k$ converges in L^m to $K_v - K_u$. Now by the convergence of b^k to b in $\mathcal{B}_p^{\gamma-1}$, we get

$$\|K_v - K_u - K_v^n + K_u^n\|_{L^m} \leq C\|b - b^n\|_{\mathcal{B}_p^{\gamma-1}}(v-u)^{1+H(\gamma-1-\frac{d}{p})}.$$

Dividing by $|v-u|^{\frac{1}{2}}$ and taking the supremum over (u, v) in $\Delta_{s,t}$ (recall that $\frac{1}{2}+H(\gamma-1-d/p) > 0$), we have

$$[K - K^n]_{\mathcal{C}^{\frac{1}{2}}_{[s,t]}L^m} \leq C\|b - b^n\|_{\mathcal{B}_p^{\gamma-1}}. \tag{4.3}$$

Bound on $E^{1,h,n}$. By Corollary 5.2, we have that for all $s < t$, $n \in \mathbb{N}$, $h \in (0, 1)$,

$$\|E_{s,t}^{1,h,n}\|_{L^m} \leq C\|b\|_{B_p^\gamma}\Big(1 + [X - B]_{\mathcal{C}^{\frac{1}{2}+H}_{[s,t]}L^m}\Big)\Big([K - K^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[s,t]}L^m} + \|\xi - \eta\|_{L^m}\Big)(t-s)^{1+H(\gamma-1-\frac{d}{p})}.$$

Since $K - K^{h,n} = K - K^n + E^{h,n}$, it comes from the previous inequality and (4.3) that

$$\|E_{s,t}^{1,h,n}\|_{L^m} \leq C\Big([E^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[s,t]}L^m} + \|b - b^n\|_{\mathcal{B}_p^{\gamma-1}} + \|\xi - \eta\|_{L^m}\Big)(t-s)^{1+H(\gamma-1-\frac{d}{p})}. \tag{4.4}$$

Bound on $E^{2,h,n}$. By Corollary 5.6, we have the following bound for $\varepsilon \in (0, \frac{1}{2})$ and $s < t$,

$$\|E_{s,t}^{2,h,n}\|_{L^m} \leq C\left(\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\right)(t-s)^{\frac{1}{2}}. \tag{4.5}$$

This is where we avoid using Girsanov's theorem and rely instead on a bound on $E^{2,h,n}$ that involves the $\mathcal{C}^1$ norm of b^n. This bound is again obtained by a stochastic sewing argument.

Conclusion. Use (4.4) and (4.5) to get

$$\begin{aligned}\|E_{s,t}^{h,n}\|_{L^m} \leq C\Big(&([E^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[s,t]}L^m} + \|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} + \|\xi - \eta\|_{L^m})(t-s)^{1+H(\gamma-1-\frac{d}{p})}\\ &+ (\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon})(t-s)^{\frac{1}{2}}\Big).\end{aligned} \tag{4.6}$$

Let $S < T \in [0, 1]$. Since $1 + H(\gamma - 1 - d/p) > 1/2$, divide by $(t-s)^{1/2}$ in (4.6) to get that for any $s < t \in [S, T]$,

$$\begin{aligned}\frac{\|E_{s,t}^{h,n}\|_{L^m}}{|t-s|^{\frac{1}{2}}} \leq C\Big(&([E^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + \|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} + \|\xi - \eta\|_{L^m})(T-S)^{\frac{1}{2}+H(\gamma-1-\frac{d}{p})}\\ &+ \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\Big).\end{aligned}$$

Taking the supremum over s, t in $[S, T]$, we get

$$\begin{aligned}[E^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\Big(&([E^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + \|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} + \|\xi - \eta\|_{L^m})(T-S)^{\frac{1}{2}+H(\gamma-1-\frac{d}{p})}\\ &+ \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\Big).\end{aligned} \tag{4.7}$$

Observe now that $[E^{h,n}]_{\mathcal{C}^{1/2}_{[S,T]}L^m}$ is finite. Indeed, in view of (4.3), the quantity $[K-K^n]_{\mathcal{C}^{1/2}_{[S,T]}L_m}$ is finite and the term $[K^n-K^{h,n}]_{\mathcal{C}^{1/2}_{[S,T]}L^m}$ is bounded by $2\|b^n\|_\infty$.

Let $\ell=(2C)^{\frac{-1}{1/2+H(\gamma-1-d/p)}}\wedge 1$. We have $1-C\ell^{1/2+H(\gamma-1-d/p)}\geq 1/2$, thus choosing $T-S\leq\ell$, we get from (4.7) that

$$[E^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m}\leq 2C\Big(\|b^n-b\|_{\mathcal{B}^{\gamma-1}_p}+\|\xi-\eta\|_{L^m}+\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon}+\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\Big).$$

Now take $S=0$ and $T=l$. Plugging the above result and (4.3) in (4.2) (with $s=0$ and $t=l$), we obtain

$$[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,l]}L^m}\leq C\Big(\|b^n-b\|_{\mathcal{B}^{\gamma-1}_p}+\|\xi-\eta\|_{L^m}+\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon}+\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\Big)\tag{4.8}$$

If $\ell=1$, the desired result is proved. Otherwise, repeat the same argument on $[\ell,2\ell\wedge 1]$, viewing $\xi\equiv X_\ell$ and $\eta\equiv X^{h,n}_\ell$ as the new initial conditions. This yields

$$\begin{aligned}[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[l,2l\wedge 1]}L^m}&\leq C\Big(\|b^n-b\|_{\mathcal{B}^{\gamma-1}_p}+\|X_\ell-X^{h,n}_\ell\|_{L^m}+\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon}+\|b^n\|_{\mathcal{C}^1}h^{1-\varepsilon}\Big)\\&\leq C\Big(\|b^n-b\|_{\mathcal{B}^{\gamma-1}_p}+\|\xi-\eta\|_{L^m}+[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,l]}L^m}+\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon}+\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\Big),\end{aligned}$$

using that $\|X_\ell-X^{h,n}_\ell\|_{L^m}\leq\|\xi-\eta\|_{L^m}+[X-X^{h,n}]_{\mathcal{C}^{1/2}_{[0,l]}L^m}$ in the last inequality. In view of (4.8), this implies

$$[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[l,2l\wedge 1]}L^m}\leq (C+C^2)\Big(\|b^n-b\|_{\mathcal{B}^{\gamma-1}_p}+\|\xi-\eta\|_{L^m}+\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon}+\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\Big).\tag{4.9}$$

Combining (4.8) and (4.9) gives a bound on $[\mathcal{E}^{h,n}]_{\mathcal{C}^{1/2}_{[0,2l\wedge 1]}L^m}$. If $2l\geq 1$, we get the desired result. Otherwise, we iterate the argument. Recalling that ℓ does not depend on h or n, we finally have

$$[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]}L^m}\leq C\Big(\|b^n-b\|_{\mathcal{B}^{\gamma-1}_p}+\|\xi-\eta\|_{L^m}+\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon}+\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\Big),\tag{4.10}$$

which concludes the proof of Theorem 2.5.

4.2 Proof of Corollary 2.6

First, let $b^n=G_{\frac{1}{n}}b$ and let n be given by $n=\lfloor h^{-\alpha}\rfloor$ for some $\alpha>0$ and $h>0$. In view of Lemma 3.2, b^n satisfies (2.10), (2.11), (2.12). Using (4.10) with $\xi=\eta$, we have

$$[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]}L^m}\leq C\Big(\lfloor h^{-\alpha}\rfloor^{-\frac{1}{2}}+\lfloor h^{-\alpha}\rfloor^{-\frac{1}{2}(\gamma-\frac{d}{p})}h^{\frac{1}{2}-\varepsilon}+\lfloor h^{-\alpha}\rfloor^{\frac{1}{2}-(\gamma-\frac{d}{p})}h^{1-\varepsilon}\Big).$$

Since $-\frac{1}{2}(\gamma-\frac{d}{p})>0$ and $\frac{1}{2}-(\gamma-\frac{d}{p})>0$, we have

$$\lfloor h^{-\alpha}\rfloor^{\frac{1}{2}-(\gamma-\frac{d}{p})}\leq h^{-\frac{\alpha}{2}}h^{\alpha(\gamma-\frac{d}{p})}\text{ and }\lfloor h^{-\alpha}\rfloor^{-\frac{1}{2}(\gamma-\frac{d}{p})}\leq h^{\frac{\alpha}{2}(\gamma-\frac{d}{p})}.$$

Moreover, since $h\in(0,\frac{1}{2})$ and $\lfloor h^{-\alpha}\rfloor>h^{-\alpha}-1$, we have

$$\begin{aligned}\lfloor h^{-\alpha}\rfloor^{-\frac{1}{2}}&\leq(1-h^{\alpha})^{-\frac{1}{2}}h^{\frac{\alpha}{2}}\\&\leq(1-\frac{1}{2^\alpha})^{-\frac{1}{2}}h^{\frac{\alpha}{2}}\leq Ch^{\frac{\alpha}{2}}.\end{aligned}$$

It follows that

$$[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]}L^m}\leq C\Big(h^{\frac{\alpha}{2}}+h^{\frac{\alpha}{2}(\gamma-\frac{d}{p})}h^{\frac{1}{2}-\varepsilon}+h^{-\frac{\alpha}{2}}h^{\alpha(\gamma-\frac{d}{p})}h^{1-\varepsilon}\Big).$$

Now we optimize over α. For the sake of readability, let $\bar{\gamma} = \gamma - \frac{d}{p} \in (1 - \frac{1}{2H}, 0)$ and introduce the following functions:

$$f_1(\alpha) = \frac{\alpha}{2}, \quad f_2(\alpha) = \frac{\bar{\gamma}}{2}\,\alpha + \frac{1}{2} \text{ and } f_3(\alpha) = (\bar{\gamma} - \frac{1}{2})\,\alpha + 1, \quad \alpha > 0.$$

Observe that f_1 is increasing and f_2, f_3 are decreasing. Moreover, we have

$$f_1(\alpha) = f_2(\alpha) = f_3(\alpha) \Leftrightarrow \alpha^\star = \frac{1}{1 - \bar{\gamma}}. \tag{4.11}$$

It follows that the error is minimized at $\alpha = \alpha^\star$. Let $n_h = h^{-\alpha^\star}$. This yields rate of convergence of order $\frac{1}{2(1-\bar{\gamma})} - \varepsilon$, which proves (2.13). Moreover, we deduce (2.14) directly from (2.13) and the following decomposition:

$$[X^{h,n_h} - B]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq [X - X^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} + [X - B]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m}.$$

4.3 Proof of Corollary 2.8

Assume that $b \in \mathcal{B}^0_\infty$. Let $\varepsilon \in (0, \frac{1}{2})$, then fix $\eta \in (0, 1/(2H) - 1)$ and $\delta \in (0, \frac{1}{2})$ such that $\eta + \delta = \varepsilon$. Then b also belongs to $\mathcal{B}^{-\eta}_\infty$ and Theorem 2.3 states that there exists a strong solution X to (1.1) which satisfies $X - B \in \mathcal{C}^{1/2+H}_{[0,T]} L^m$, which is pathwise unique in the class of solutions that satisfy $X - B \in \mathcal{C}^{1/2+H}_{[0,T]} L^2$.

To prove the second part of the corollary, apply Theorem 2.5 with $\gamma = -\eta$, $p = \infty$ and $\varepsilon = \delta$, and note that $\|b^n - b\|_{\mathcal{B}^{-\eta-1}_\infty} \leq C \|b^n - b\|_{\mathcal{B}^{-1}_\infty}$ to get that

$$[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C \left(\|b^n - b\|_{\mathcal{B}^{-1}_\infty} + \|b^n\|_\infty h^{\frac{1}{2} - \delta} + \|b^n\|_{\mathcal{C}^1} \|b^n\|_\infty h^{1-\delta} \right).$$

Now we take $b^n = G_{\frac{1}{n}} b$ and $n = \lfloor h^{-\alpha} \rfloor$ for some $\alpha > 0$. Using (2.10), (2.11) and (2.12) as in Subsection 4.2, this leads to

$$[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C \left(h^{\frac{\alpha}{2}} + h^{\frac{1}{2} - \delta - \frac{\alpha\eta}{2}} + h^{1 - \delta - \frac{\alpha}{2} - \alpha\eta} \right).$$

Optimising over α as before, we find $\alpha^\star = 1/(1 + \eta)$, which yields a rate of convergence of order $\frac{1}{2(1+\eta)} - \delta$. Since $\frac{1}{2(1+\eta)} \geq 1/2 - \eta$, it finally comes that

$$[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C\, h^{\frac{1}{2} - \eta - \delta} = C\, h^{\frac{1}{2} - \varepsilon}.$$

5 Regularisation effect of fBm and discrete-time fBm

In this section, X always denotes a weak solution to (1.1) with drift $b \in \mathcal{B}^\gamma_p$, with $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$ satisfying (2.7). For such X, recall that the process K is defined by (2.4). Let $(b^n)_\mathbb{N}$ a be sequence of smooth functions that converges to b in $\mathcal{B}^{\gamma-}_p$. For $n \in \mathbb{N}$ and $h \in (0, 1)$, $X^{h,n}$ denotes the Euler scheme (1.2), and recall that the process $K^{h,n}$ is defined by (4.1). As in Section 4.1, we allow different random initial conditions for the solution and the Euler scheme (respectively denoted ξ and η).

5.1 Regularisation in the strong well-posedness regime

In this subsection, we state and prove the bound on $E^{1,h,n}$. The following proposition is close to Proposition 3.5 but it requires the stronger assumption (2.7).

Proposition 5.1. *Let $(\psi_t)_{t\in[0,1]}$, $(\phi_t)_{t\in[0,1]}$ be two $\mathbb{R}^d$-valued stochastic processes adapted to $\mathbb{F}$. Let $f \in \mathcal{C}_b^\infty(\mathbb{R}^d,\mathbb{R}^d)\cap\mathcal{B}_p^\gamma$, $m\geq 2$, and let $\tau\in(0,1)$ be such that*

$$\tau + H(\gamma - 1 - \frac{d}{p}) > 0. \tag{5.1}$$

There exists a constant $C>0$ such that for any $(s,t)\in\Delta_{0,1}$,

$$\begin{aligned}&\Big\| \int_s^t (f(\psi_r+B_r)-f(\phi_r+B_r))\,dr\Big\|_{L^m}\\ &\quad\leq C\,\|f\|_{\mathcal{B}_p^\gamma}\Big(1+[\psi]_{\mathcal{C}^{\frac12+H}_{[s,t]}L^m}\Big)\Big([\psi-\phi]_{\mathcal{C}^\tau_{[s,t]}L^m}+\|\psi_s-\phi_s\|_{L^m}\Big)(t-s)^{1+H(\gamma-1-\frac{d}{p})}.\end{aligned}\tag{5.2}$$

Proof. Let $0\leq S<T\leq 1$. For $(s,t)\in\Delta_{S,T}$, let

$$A_{s,t}=\int_s^t\Big(f(\psi_s+B_r)-f(\phi_s+B_r)\Big)\,dr \quad\text{and}\quad \mathcal{A}_t=\int_S^t\Big(f(\psi_r+B_r)-f(\phi_r+B_r)\Big)\,dr. \tag{5.3}$$

Assume without any loss of generality that $[\psi]_{\mathcal{C}^{1/2+H}_{[S,T]}L^m}$ and $[\psi-\phi]_{\mathcal{C}^\tau_{[S,T]}L^m}$ are finite, otherwise the result is trivial. In the following, we check the conditions in order to apply Lemma A.1. To show that (A.1) and (A.2) hold true with $\varepsilon_1=\tau\wedge\frac12+H(\gamma-1-d/p)>0$ and $\varepsilon_2=1/2+H(\gamma-d/p)>0$, we prove that there exists a constant $C>0$ independent of s,t,s and t such that for $u=(s+t)/2$,

(i) $\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m}\leq C\,\|f\|_{\mathcal{B}_p^\gamma}([\psi]_{\mathcal{C}^{1/2+H}_{[S,T]}L^m}+1)([\psi-\phi]_{\mathcal{C}^\tau_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})(t-s)^{1+\tau\wedge\frac12+H(\gamma-1-d/p)}$;

(ii) $\|\delta A_{s,u,t}\|_{L^m}\leq C\,\|f\|_{\mathcal{B}_p^\gamma}([\psi-\phi]_{\mathcal{C}^\tau_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})(t-s)^{1+H(\gamma-1-\frac{d}{p})}$;

(iii) If (i) and (ii) are satisfied, (A.3) gives the convergence in probability of $\sum_{i=1}^{N_n-1}A_{t_i^k,t_{i+1}^k}$ along any sequence of partitions $\Pi_k=\{t_i^k\}_{i=1}^{N_k}$ of $[S,t]$ with mesh converging to 0. We will prove that the limit is the process $\mathcal{A}$ given in (5.3).

Assume for now that (i), (ii) and (iii) hold. Applying Lemma A.1, we obtain that

$$\begin{aligned}\Big\|\int_s^t f(B_r+\psi_r)\,dr\Big\|_{L^m}&\leq C\,\|f\|_{\mathcal{B}_p^\gamma}([\psi-\phi]_{\mathcal{C}^\tau_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})(t-s)^{1+H(\gamma-1-\frac{d}{p})}\\ &\quad+C\,\|f\|_{\mathcal{B}_p^\gamma}([\psi]_{\mathcal{C}^{1/2+H}_{[S,T]}L^m}+1)([\psi-\phi]_{\mathcal{C}^\tau_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})(t-s)^{1+\tau\wedge\frac12+H(\gamma-1-d/p)}\\ &\quad+\|A_{s,t}\|_{L^m}.\end{aligned}$$

We will see in (5.5) that $\|A_{s,t}\|_{L^m}\leq C\,\|f\|_{\mathcal{B}_p^\gamma}([\psi-\phi]_{\mathcal{C}^\tau_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})(t-s)^{1+H(\gamma-1-\frac{d}{p})}$. Then, choosing $(s,t)=(S,T)$ we get (5.2).

We now check that the conditions (i), (ii) and (iii) actually hold.

Proof of (i): For $u\in[s,t]$, by the tower property of conditional expectation, we have

$$\begin{aligned}\mathbb{E}^s\delta A_{s,u,t}&=\mathbb{E}^s\int_u^t\mathbb{E}^u\Big[f(\psi_s+B_r)-f(\phi_s+B_r)-f(\psi_u+B_r)+f(\phi_u+B_r)\Big]\,dr\\ &=:\mathbb{E}^s\int_u^t\mathbb{E}^u[F(B_r,s,u)+G(B_r,s,u)]\,dr,\end{aligned}\tag{5.4}$$

where

$$\begin{aligned}F(\cdot,s,u)&=f(\psi_s+\cdot)-f(\phi_s+\cdot)-f(\psi_u+\cdot)+f(\psi_u+\phi_s-\psi_s+\cdot),\\ K(\cdot,s,u)&=f(\psi_u+\phi_s-\psi_s+\cdot)-f(\phi_u+\cdot).\end{aligned}$$

By Lemma 3.3(ii), we have that

$$|\mathbb{E}^u F(B_r,s,u)|\leq \|F(\cdot,s,u)\|_{\mathcal{B}_p^{\gamma-2}}(r-u)^{H(\gamma-2-\frac{d}{p})},$$
$$|\mathbb{E}^u K(B_r,s,u)|\leq \|K(\cdot,s,u)\|_{\mathcal{B}_p^{\gamma-1}}(r-u)^{H(\gamma-1-\frac{d}{p})}.$$

Moreover, by Lemma 3.1(iii), it comes that

$$\begin{aligned}\|\|F(\cdot,s,u)\|_{\mathcal{B}_p^{\gamma-2}}\|_{L^m} &\leq \|f\|_{\mathcal{B}_p^{\gamma}}\|\psi_s-\psi_u\|_{L^m}\|\psi_s-\phi_s\|_{L^m}\\ &\leq C\,\|f\|_{\mathcal{B}_p^{\gamma}}\,[\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}(u-s)^{\frac12+H}\|\psi-\phi\|_{L^\infty_{[S,T]}L^m}\\ &\leq C\,\|f\|_{\mathcal{B}_p^{\gamma}}\,[\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}([\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})(u-s)^{\frac12+H},\end{aligned}$$

and

$$\begin{aligned}\|\|K(\cdot,s,u)\|_{\mathcal{B}_p^{\gamma-1}}\|_{L^m} &\leq \|f\|_{\mathcal{B}_p^{\gamma}}\|\psi_s-\psi_u-\phi_s+\phi_u\|_{L^m}\\ &\leq C\,\|f\|_{\mathcal{B}_p^{\gamma}}\,[\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]}L^m}\,(u-s)^{\tau}.\end{aligned}$$

Plugging the previous bounds in (5.4), we obtain

$$\begin{aligned}\|\mathbb{E}^s\delta A_{s,u,t}\|_{L^m} &\leq C\,\|f\|_{\mathcal{B}_p^{\gamma}}\Big([\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}+1\Big)([\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})\\ &\qquad\times\Big((t-s)^{1+\tau+H(\gamma-1-\frac{d}{p})}+(t-s)^{1+\frac12+H+H(\gamma-2-\frac{d}{p})}\Big)\\ &\leq C\,\|f\|_{\mathcal{B}_p^{\gamma}}\,\Big([\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}+1\Big)([\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})(t-s)^{1+\tau\wedge\frac12+H(\gamma-1-\frac{d}{p})}.\end{aligned}$$

Proof of (ii): we write

$$\|\delta A_{s,u,t}\|_{L^m}\leq \|A_{s,t}\|_{L^m}+\|A_{s,u}\|_{L^m}+\|A_{u,t}\|_{L^m}$$

and we will apply Lemma 3.4 for each term in the right-hand side of the previous inequality, respectively for $\Xi=(\psi_s,\phi_s)$, (ψ_s,ϕ_s) again and (ψ_u,ϕ_u). We only detail the first one. As f is smooth and bounded, the first assumption of Lemma 3.4 is verified. By Lemma 3.1(i), $\|f(\cdot+\psi_s)-f(\cdot+\phi_s)\|_{\mathcal{B}_p^{\gamma}}\leq 2\|f\|_{\mathcal{B}_p^{\gamma}}$, hence the second assumption of Lemma 3.4 is verified. It follows that

$$\begin{aligned}\|A_{s,t}\|_{L^m} &\leq C\,\|\|f(\psi_s+\cdot)-f(\phi_s+\cdot)\|_{\mathcal{B}_p^{\gamma-1}}\|_{L^m}\,(t-s)^{1+H(\gamma-1-\frac{d}{p})}\\ &\leq C\,\|f\|_{\mathcal{B}_p^{\gamma}}\,\|\psi_s-\phi_s\|_{L^m}\,(t-s)^{1+H(\gamma-1-\frac{d}{p})}\\ &\leq C\,\|f\|_{\mathcal{B}_p^{\gamma}}([\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})\,(t-s)^{1+H(\gamma-1-\frac{d}{p})}.\end{aligned}\tag{5.5}$$

Hence combining similar inequalities on $A_{s,u}$ and $A_{u,t}$ with (5.5), we get

$$\|\delta A_{s,u,t}\|_{L^m}\leq C\,\|f\|_{\mathcal{B}_p^{\gamma}}\,([\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})\,(t-s)^{1+H(\gamma-1-\frac{d}{p})}.$$

Proof of (iii): Finally, for a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[s,t]$ with $\Pi_k=\{t_i^k\}_{i=1}^{N_k}$ and mesh size $|\Pi_k|$ converging to zero, we have

$$\begin{aligned}\|\mathcal{A}_t-\sum_{i=1}^{N_k-1}A_{t_i^k,t_{i+1}^k}\|_{L^m} &\leq \sum_i\int_{t_i^k}^{t_{i+1}^k}\Big\|f(\psi_r+B_r)-f(\phi_r+B_r)-f(\psi_{t_i^k}+B_r)+f(\phi_{t_i^k}+B_r)\Big\|_{L^m}\,dr\\ &\leq C\,\|f\|_{\mathcal{C}^2}\sum_i\int_{t_i^k}^{t_{i+1}^k}\Big(|\Pi_k|^{\tau}[\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]}L^m}+|\Pi_k|^{\frac12+H}[\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}([\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]}L^m}+\|\psi_S-\phi_S\|_{L^m})\Big)\,dr,\end{aligned}$$

where we used again the decomposition with the functions F and G defined above in the proof. Hence

$$\begin{aligned}
&\|\mathcal{A}_t - \sum_{i=1}^{N_k-1} A_{t_i^k, t_{i+1}^k}\|_{L^m} \\
&\quad \le C\,\|f\|_{\mathcal{C}^2}\, t\left(|\Pi_k|^\tau [\psi-\phi]_{\mathcal{C}^\tau_{[S,T]}L^m} + |\Pi_k|^{\frac{1}{2}+H} [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^m} ([\psi-\phi]_{\mathcal{C}^\tau_{[S,T]}L^m} + \|\psi_S-\phi_S\|_{L^m})\right) \\
&\quad \underset{k\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

□

Corollary 5.2. *Recall that the process $K^{h,n}$ was defined in (4.1). Let $m \ge 2$. There exists a constant $C > 0$ such that for any $0 \le S < T \le 1$, any $\mathcal{F}_S$-measurable random variables ξ and η and any $(s,t) \in \Delta_{S,T}$, we have*

$$\begin{aligned}
&\Big\| \int_s^t (b^n(\xi + K_r + B_r) - b^n(\eta + K_r^{h,n} + B_r))\, dr \Big\|_{L^m} \\
&\quad \le C\|b^n\|_{\mathcal{B}_p^\gamma}\Big(1 + [X-B]_{\mathcal{C}^{\frac{1}{2}+H}_{[s,t]}L^m}\Big)\Big([K-K^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[s,t]}L^m} + \|\xi-\eta\|_{L^m}\Big)(t-s)^{1+H(\gamma-1-\frac{d}{p})}.
\end{aligned}$$

Proof. Choose $\tau = \frac{1}{2}$, which satisfies (5.1) since in this case, it is reduced to (2.7). Apply Proposition 5.1 with $\tau = \frac{1}{2}$, $f = b^n$, $\psi = \xi + K$ and $\phi = \eta + K^{h,n}$. Also recall from (2.3) that $\|b^n\|_{\mathcal{B}_p^\gamma} \le \|b\|_{\mathcal{B}_p^\gamma}$ to get

$$\begin{aligned}
&\Big\| \int_s^t (b^n(K_r + x + B_r) - b^n(K_r^{h,n} + y + B_r))\, dr \Big\|_{L^m} \\
&\quad \le C\|b\|_{\mathcal{B}_p^\gamma}\Big(1 + [K]_{\mathcal{C}^{\frac{1}{2}+H}_{[s,t]}L^m}\Big)\Big([K-K^{h,n}]_{\mathcal{C}^{\frac{1}{2}}[s,t]L^m} + \|\xi-\eta\|_{L^m}\Big)(t-s)^{1+H(\gamma-1-\frac{d}{p})}.
\end{aligned}$$

□

5.2 Sewing bounds for the d-dimensional discrete fBm

In this section, we present general Lemmas that lead to a bound on the term $E^{2,h,m}$.

Lemma 5.3. *Let $f \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\gamma$ and recall that γ and p satisfy (2.7). Let $\varepsilon \in (0, \frac{1}{2})$, $m \in [2, \infty)$ and $h > 0$. There exists a constant $C > 0$ such that for any $0 \le S < T \le 1$, any $\mathbb{R}^d$-valued $\mathcal{F}_S$-measurable random variable ψ and any $(s,t) \in \Delta_{S,T}$, we have*

$$\Big\| \int_s^t f(\psi + B_r) - f(\psi + B_{r_h})\, dr \Big\|_{L^m} \le C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

Proof. We will check the conditions in order to apply Lemma A.1. For $(s,t) \in \Delta_{S,T}$, let

$$A_{s,t} = \mathbb{E}^s \int_s^t f(\psi + B_r) - f(\psi + B_{r_h})\, dr \quad\text{and}\quad \mathcal{A}_t = \int_S^t f(\psi + B_r) - f(\psi + B_{r_h})\, dr.$$

Let $u \in [s,t]$ and notice that $\mathbb{E}^s \delta A_{s,u,t} = 0$, so (A.1) holds with $\Gamma_1 = 0$. We will prove that (A.2) holds with

$$\Gamma_2 = C\|f\|_\infty h^{\frac{1}{2}-\varepsilon}.$$

The case $t - s \le 2h$. In this case we have

$$|A_{s,t}| \le \|f\|_\infty (t-s) \le \|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\varepsilon}. \tag{5.6}$$

The case $t-s>2h$. Here we split $A_{s,t}$ in two

$$A_{s,t}=\mathbb{E}^s\int_s^{s+2h} f(\psi+B_r)-f(\psi+B_{r_h})\,dr+\mathbb{E}^s\int_{s+2h}^t f(\psi+B_r)-f(\psi+B_{r_h})\,dr.$$

For the first part, we obtain

$$\left|\mathbb{E}^s\int_s^{s+2h} f(\psi+B_r)-f(\psi+B_{r_h})\,dr\right|\leq 2h\,\|f\|_\infty\leq C\,\|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\varepsilon}.$$

Denote the second part as

$$J:=\int_{s+2h}^t \mathbb{E}^s[f(\psi+B_r)-f(\psi+B_{r_h})]\,dr.$$

We write $B_r=B_r-\mathbb{E}^sB_r+\mathbb{E}^sB_r$, where $B_r-\mathbb{E}^sB_r$ is a Gaussian variable independent of $\mathcal{F}_s$ and of variance $C(r-s)^{2H}$, see [6, Proposition 3.6 (ii)]. Using the Gaussian semigroup G, we get

$$\begin{aligned} J&=\int_{s+2h}^t \Big(G_{C(r-s)^{2H}}f(\psi+\mathbb{E}^sB_r)-G_{C(r_h-s)^{2H}}f(\psi+\mathbb{E}^sB_{r_h})\Big)\,dr \qquad (5.7)\\ &=\int_{s+2h}^t \Big(G_{C(r-s)^{2H}}f(\psi+\mathbb{E}^sB_r)-G_{C(r_h-s)^{2H}}f(\psi+\mathbb{E}^sB_r)\Big)\,dr\\ &\quad+\int_{s+2h}^t G_{C(r-s)^{2H}}\Big[f(\psi+\mathbb{E}^sB_r)-f(\psi+\mathbb{E}^sB_{r_h})\Big]\,dr\\ &=:J_1+J_2.\end{aligned}$$

For J_1, we apply [6, Proposition 3.7 (ii)] with $\beta=0$, $\delta=1$, $\alpha=0$ to get

$$\|J_1\|_{L^m}\leq C\,\|f\|_\infty\int_{s+2h}^t((r-s)^{2H}-(r_h-s)^{2H})(r_h-s)^{-2H}\,dr.$$

Now applying the inequalities $(r-s)^{2H}-(r_h-s)^{2H}\leq C(r-r_h)(r_h-s)^{2H-1}$ and $2(r_h-s)\geq(r-s)$, it comes

$$\begin{aligned}\|J_1\|_{L^m}&\leq C\,\|f\|_\infty\int_{s+2h}^t(r-r_h)(r-s)^{2H-1}(r-s)^{-2H}\,dr\\ &\leq C\,\|f\|_\infty h\int_{s+2h}^t(r-s)^{-1}\,dr\\ &\leq C\,\|f\|_\infty h\,(|\log(2h)|+|\log(t-s)|)\,.\end{aligned}$$

Use again that $2h<t-s$ to get

$$\|J_1\|_{L^m}\leq C\,\|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

As for J_2, we have

$$\|J_2\|_{L^m}\leq\int_{s+2h}^t\|G_{C(r-s)^{2H}}f\|_{\mathcal{C}^1}\,\|\mathbb{E}^sB_r-\mathbb{E}^sB_{r_h}\|_{L^m}\,dr.$$

In view of [6, Proposition 3.7 (i)] applied with $\beta=1$, $\alpha=0$ and [6, Proposition 3.6 (v)], we get

$$\begin{aligned}\|J_2\|_{L^m}&\leq C\,\|f\|_\infty\int_{s+2h}^t\|\mathbb{E}^sB_r-\mathbb{E}^sB_{r_h}\|_{L^m}(r-s)^{-H}\,dr\\ &\leq C\,\|f\|_\infty\int_{s+2h}^t(r-r_h)(r-s)^{H-1}(r-s)^{-H}\,dr\\ &\leq C\,\|f\|_\infty\,h\,(|\log(2h)|+|\log(t-s)|)\\ &\leq C\,\|f\|_\infty\,h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.\end{aligned}$$

Combining the bounds on J_1 and J_2, we deduce that

$$\|J\|_{L^m}\leq C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

Hence for all $t-s>2h$,

$$\|A_{s,t}\|_{L^m}\leq C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}. \tag{5.8}$$

Overall, combining (5.6) and (5.8), we obtain that for all $s\leq t$,

$$\|A_{s,t}\|_{L^m}\leq C\|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

Thus for any $u\in[s,t]$,

$$\begin{aligned}\|\delta A_{s,u,t}\|_{L^m} &\leq \|A_{s,t}\|_{L^m}+\|A_{s,u}\|_{L^m}+\|A_{u,t}\|_{L^m}\\ &\leq C\|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.\end{aligned}$$

The power in $(t-s)$ is strictly larger than $1/2$, so (A.2) holds.

Convergence in probability. Finally, for a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[S,t]$ with $\Pi_k=\{t_i^k\}_{i=1}^{N_k}$ and mesh size converging to zero, we have

$$\begin{aligned}\Big\|\mathcal{A}_t-\sum_{i=1}^{N_k-1}A_{t_i^k,t_{i+1}^k}\Big\|_{L^1} &\leq \sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\mathbb{E}\Big|f(\psi+B_r)-f(\psi+B_{r_h})-\mathbb{E}^{t_i^k}[f(\psi+B_r)+f(\psi-B_{r_h})]\Big|\,dr\\ &\leq \sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\mathbb{E}\Big|f(\psi+B_r)-\mathbb{E}^{t_i^k}f(\psi+B_r)\Big|\,dr\\ &\quad+\sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\mathbb{E}\Big|f(\psi+B_{r_h})-\mathbb{E}^{t_i^k}f(\psi+B_{r_h})\Big|\,dr\\ &=: I_1+I_2.\end{aligned}$$

In view of Lemma 3.3(iii), it comes that

$$I_1\leq\sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\|f\|_{\mathcal{C}^1}(r-t_i^k)^H\,dr\leq\|f\|_{\mathcal{C}^1}|\Pi_k|^H\,(t-S).$$

As for I_2, note that if $r_h\leq t_i^k$, then $\mathbb{E}|f(\psi+B_{r_h})-\mathbb{E}^{t_i^k}f(\psi+B_{r_h})|=0$. On the other hand, when $r_h\in(t_i^k,t_{i+1}^k]$ then in view of Lemma 3.3(iii), we have

$$\mathbb{E}|f(\psi+B_{r_h})-\mathbb{E}^{t_i^k}f(\psi+B_{r_h})|\leq C\|f\|_{\mathcal{C}^1}|\Pi_k|^H.$$

It follows that

$$I_2\leq\sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\|f\|_{\mathcal{C}^1}|\Pi_k|^H\,dr,$$

and therefore $\sum_{i=1}^{N_k-1}A_{t_i^k,t_{i+1}^k}$ converges in probability to $\mathcal{A}_t$ as $k\to+\infty$. We can therefore apply Lemma A.1 with $\varepsilon_1>0$ and $\varepsilon_2=\varepsilon/2$ to conclude that

$$\begin{aligned}\|\mathcal{A}_t-\mathcal{A}_s\|_{L^m} &\leq \|\mathcal{A}_t-\mathcal{A}_s-A_{s,t}\|_{L^m}+\|A_{s,t}\|_{L^m}\\ &\leq C\,\|f\|_\infty\,h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.\end{aligned}$$

□

Proposition 5.4. *Let $(\psi_t)_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued stochastic process adapted to $\mathbb{F}$. Let $f \in \mathcal{C}_b^\infty(\mathbb{R}^d,\mathbb{R}^d)\cap \mathcal{B}_p^\gamma$ and recall that γ and p satisfy (2.7). Let $\varepsilon \in (0,\frac{1}{2})$ and $m\in[2,\infty)$. There exists a constant $C>0$ such that for any $(s,t)\in\Delta_{0,1}$, we have*

$$\Big\| \int_s^t f(\psi_r + B_r) - f(\psi_r + B_{r_h})\, dr \Big\|_{L^m} \le C\Big(\|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}} + \|f\|_{\mathcal{C}^1}[\psi]_{\mathcal{C}^1_{[s,t]}L^m} h^{1-\varepsilon}(t-s)^{1+\frac{\varepsilon}{2}}\Big). \tag{5.9}$$

Proof. We will check the conditions in order to apply Lemma A.1. Let and $0\le S<T\le 1$. For any $(s,t)\in\Delta_{S,T}$, define

$$A_{s,t} = \int_s^t f(\psi_s + B_r) - f(\psi_s + B_{r_h})\, dr \ \text{ and } \ \mathcal{A}_t = \int_S^t f(\psi_r + B_r) - f(\psi_r + B_{r_h})\, dr.$$

To show that (A.1) and (A.2) hold true with $\varepsilon_1=\varepsilon_2=\varepsilon/2>0$, we prove that there exists a constant $C>0$ independent of s,t,S and T such that for $u=(s+t)/2$,

(i) $\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \le C\, \|f\|_{\mathcal{C}^1}\, [\psi]_{\mathcal{C}^1_{[S,T]}L^m}\, h^{1-\varepsilon}\, (t-s)^{1+\frac{\varepsilon}{2}}$;

(ii) $\|\delta A_{s,u,t}\|_{L^m} \le C\, \|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\, (t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}$;

(iii) If (i) and (ii) are satisfied, (A.3) gives the convergence in probability of $\sum_{i=1}^{N_n-1} A_{t_i^k, t_{i+1}^k}$ along any sequence of partitions $\Pi_k=\{t_i^k\}_{i=1}^{N_k}$ of $[S,t]$ with mesh converging to 0. We will prove that the limit is the process $\mathcal{A}$ given in (5.3).

Assume for now that (i), (ii) and (iii) hold. Applying Lemma A.1, we obtain that

$$\Big\| \int_s^t f(\psi_r + B_r) - f(\psi_r + B_{r_h})\, dr \Big\|_{L^m} \le C\Big(\|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}} + \|f\|_{\mathcal{C}^1}[\psi]_{\mathcal{C}^1_{[s,t]}L^m} h^{1-\varepsilon}(t-s)^{1+\frac{\varepsilon}{2}}\Big) + \|A_{s,t}\|_{L^m}.$$

We will see in (5.13) that $\|A_{s,t}\|_{L^m}\le C\,\|f\|_\infty h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}$. Then choosing $(s,t)=(S,T)$, we get (5.9).

We now check that the conditions (i), (ii) and (iii) actually hold.

Proof of (i): For $u\in[s,t]$, by the tower property of conditional expectation, we have

$$\mathbb{E}^s \delta A_{s,u,t} = \mathbb{E}^s \int_u^t \mathbb{E}^u[f(\psi_s + B_r) - f(\psi_s + B_{r_h}) - f(\psi_u + B_r) + f(\psi_u + B_{r_h})]\, dr.$$

The case $u-s\le 2h$. In this case, using the Lipschitz norm of f, we have

$$|\mathbb{E}^s\delta A_{s,u,t}| \le 2\|f\|_{\mathcal{C}^1}\int_u^t |\psi_s-\psi_u|\, dr,$$

and therefore using the inequality $(t-u)(u-s)\le (t-u)(u-s)^\varepsilon h^{1-\varepsilon} \le (t-s)^{1+\varepsilon}h^{1-\varepsilon}$,

$$\begin{aligned}\|\mathbb{E}^s\delta A_{s,u,t}\|_{L^m} &\le \|f\|_{\mathcal{C}^1}\|\psi\|_{\mathcal{C}^1_{[S,T]}L^m}(t-u)(u-s)\\ &\le C\|f\|_{\mathcal{C}^1}\|\psi\|_{\mathcal{C}^1_{[S,T]}L^m}(t-s)^{1+\varepsilon}h^{1-\varepsilon}.\end{aligned}$$

The case $u-s>2h$. We split the integral first between u and $u+2h$ and then between $u+2h$ and t as

$$\delta A_{s,u,t} =: J_1 + J_2.$$

For J_1, we obtain as before that

$$|\mathbb{E}^s \delta A_{s,u,u+2h}| \leq 2\|f\|_{\mathcal{C}^1} |\psi_s - \psi_u| \,(u + 2h - u),$$

and using the fact that $u - s > 2h$, it follows that

$$\|\mathbb{E}^s J_1\|_{L^m} = \|\mathbb{E}^s \delta A_{s,u,u+2h}\|_{L^m} \leq C\|f\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1_{[S,T]} L^m} (t-s)^{1+\varepsilon} h^{1-\varepsilon}. \tag{5.10}$$

As for J_2, we write as in (5.7) that

$$\begin{aligned}
J_2 &= \mathbb{E}^s \int_{u+2h}^t \mathbb{E}^u[f(\psi_s + B_r) - f(\psi_s + B_{r_h}) - f(\psi_u + B_r) + f(\psi_u + B_{r_h})]\, dr \\
&= \mathbb{E}^s \int_{u+2h}^t (G_{C(r-u)^{2H}} - G_{C(r_h-u)^{2H}})(f(\psi_s + \mathbb{E}^u B_r) - f(\psi_u + \mathbb{E}^u B_r))\, dr \\
&\quad + \mathbb{E}^s \int_{u+2h}^t G_{C(r_h-u)^{2H}} \Big(f(\psi_s + \mathbb{E}^u B_r) - f(\psi_s + \mathbb{E}^u B_{r_h}) - f(\psi_u + \mathbb{E}^u B_r) + f(\psi_u + \mathbb{E}^u B_{r_h})\Big)\, dr \\
&=: J_{21} + J_{22}.
\end{aligned}$$

For J_{21}, we apply [6, Proposition 3.7 (ii)] with $\beta = 0$, $\delta = 1$, $\alpha = 0$ to get

$$\|J_{21}\|_{L^m} \leq C\| \|f(\psi_s + \cdot) - f(\psi_u + \cdot)\|_\infty \|_{L^m} \int_{u+2h}^t ((r-s)^{2H} - (r_h - s)^{2H})(r_h - s)^{-2H}\, dr.$$

Now applying the inequalities $(r-s)^{2H} - (r_h-s)^{2H} \leq C(r-r_h)(r_h-s)^{2H-1}$ and $2(r_h-s) \geq (r-s)$, it comes

$$\begin{aligned}
\|J_{21}\|_{L^m} &\leq C\, \|f\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1_{[S,T]} L^m} |u-s| \int_{u+2h}^t (r - r_h)(r-s)^{2H-1}(r-s)^{-2H}\, dr \\
&\leq C\, \|f\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1_{[S,T]} L^m}\, h\,(t-s)\,(|\log(2h)| + |\log(t-s)|).
\end{aligned}$$

Since $2h \leq t - s$, one has

$$\|J_{21}\|_{L^m} \leq C\, \|f\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1_{[S,T]} L^m}\, h^{1-\varepsilon} (t-s)^{1+\frac{\varepsilon}{2}}. \tag{5.11}$$

As for J_{22}, observe that

$$\begin{aligned}
&\Big| G_{C(r_h-u)^{2H}} \Big(f(\psi_s + \mathbb{E}^u B_r) - f(\psi_s + \mathbb{E}^u B_{r_h}) - f(\psi_u + \mathbb{E}^u B_r) + f(\psi_u + \mathbb{E}^u B_{r_h})\Big) \Big| \\
&\quad \leq \|G_{C(r_h-u)^{2H}}(f(\psi_s + \cdot) - f(\psi_u + \cdot))\|_{\mathcal{C}^1}\, |\mathbb{E}^u B_r - \mathbb{E}^u B_{r_h}| \\
&\quad \leq C\, \|f(\psi_s + \cdot) - f(\psi_u + \cdot)\|_\infty\, (r_h - u)^{-H}\, |\mathbb{E}^u B_r - \mathbb{E}^u B_{r_h}|,
\end{aligned}$$

where we used [6, Proposition 3.7 (i)] with $\beta = 1$ and $\alpha = 0$ in the last inequality. Now in view of [6, Proposition 3.6 (v)], the previous inequality and using $2(r_h - u) \geq r - u$, it comes

$$\begin{aligned}
\|J_{22}\|_{L^m} &\leq C\, \| \|f(\psi_s + \cdot) - f(\psi_u + \cdot)\|_\infty \|_{L^m} \int_{u+2h}^t \|\mathbb{E}^u B_r - \mathbb{E}^u B_{r_h}\|_{L^m} (r_h - u)^{-H}\, dr \\
&\leq C\, \|f\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1_{[S,T]} L^m}\, (u-s) \int_{u+2h}^t (r - r_h)(r-u)^{H-1}(r_h - u)^{-H}\, dr \\
&\leq C\, \|f\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1_{[S,T]} L^m}\, h\,(t-s)\,(|\log(2h)| + |\log(t-s)|) \\
&\leq C\, \|f\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1_{[S,T]} L^m}\, h^{1-\varepsilon}\,(t-s)^{1+\frac{\varepsilon}{2}}.
\end{aligned} \tag{5.12}$$

In view of the inequalities (5.10), (5.11) and (5.12), we have finally

$$\|\mathbb{E}^s \delta A_{s,u,t}\|_{L^{p'}} \leq C\, \|f\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1_{[S,T]} L^m} h^{1-\varepsilon} (t-s)^{1+\frac{\varepsilon}{2}}.$$

Proof of (ii): we write

$$\|\delta A_{s,u,t}\|_{L^m}\leq \|A_{s,t}\|_{L^m}+\|A_{s,u}\|_{L^m}+\|A_{u,t}\|_{L^m}$$

and we apply Lemma 5.3 for each term in the right-hand side of the previous inequality, respectively for $\psi = \psi_s$, ψ_s again and ψ_u. We thus have

$$\|A_{s,t}\|_{L^m}\leq C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}, \tag{5.13}$$

and combining similar inequalities on $A_{s,u}$ and $A_{u,t}$ with (5.13) yields

$$\|\delta A_{s,u,t}\|_{L^m}\leq C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

Proof of (iii): finally, for a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[S,t]$ with $\Pi_k=\{t_i^k\}_{i=1}^{N_k}$ and mesh size $|\Pi_k|$ converging to zero, we have

$$\begin{aligned}\|\mathcal{A}_t-\sum_{0}^{N-1}A_{t_i,t_{i+1}}\|_{L^1} &\leq \sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\mathbb{E}|f(\psi_r+B_r)-f(\psi_r+B_{r_h})-f(\psi_{t_i^k}+B_r)+f(\psi_{t_i^k}+B_{r_h})|\,dr\\ &\leq C\sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\|f\|_{\mathcal{C}^1}\|\psi_r-\psi_{t_i^k}\|_{L^1}\,dr\\ &\leq C\|f\|_{\mathcal{C}^1}\sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\|\psi\|_{\mathcal{C}^1_{[0,1]}L^m}|\Pi_k|\,dr\underset{k\to\infty}{\longrightarrow}0.\end{aligned}$$

□

Corollary 5.5. *Let $(\psi_t)_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued stochastic process adapted to $\mathbb{F}$. Let $f\in\mathcal{C}_b^\infty(\mathbb{R}^d,\mathbb{R}^d)\cap\mathcal{B}_p^\gamma$ and recall that γ and p satisfy (2.7). Let $\varepsilon\in(0,\frac{1}{2})$ and $m\in[2,\infty)$. There exists a constant $C>0$ such that for any $(s,t)\in\Delta_{0,1}$,*

$$\Big\|\int_s^t f(\psi_r+B_r)-f(\psi_{r_h}+B_{r_h})\,dr\Big\|_{L^m}\leq C\left(\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}+\|f\|_{\mathcal{C}^1}[\psi]_{\mathcal{C}^1_{[0,1]}L^m}\,h^{1-\varepsilon}(t-s)\right).$$

Proof. Introducing the pivot term $f(\psi_r+B_{r_h})$, we have

$$\begin{aligned}&\Big\|\int_s^t f(\psi_r+B_r)-f(\psi_{r_h}+B_{r_h})\,dr\Big\|_{L^m}\\ &\leq \Big\|\int_s^t f(\psi_{r_h}+B_{r_h})-f(\psi_r+B_{r_h})\,dr\Big\|_{L^m}+\Big\|\int_s^t f(\psi_r+B_r)-f(\psi_r+B_{r_h})\,dr\Big\|_{L^m}\\ &=: J_1+J_2.\end{aligned}$$

We first bound J_1 using the $\mathcal{C}^1$ norm of f:

$$J_1\leq \|f\|_{\mathcal{C}^1}\int_s^t\|\psi_r-\psi_{r_h}\|_{L^m}\,dr\leq \|f\|_{\mathcal{C}^1}\|\psi\|_{\mathcal{C}^1_{[0,1]}L^m}\,h\,(t-s).$$

Then J_2 is bounded by Proposition 5.4. Combining the two bounds, we get the desired result. □

Corollary 5.6. *Recall the process $K^{h,n}$ that was defined in (4.1). Let $\varepsilon\in(0,\frac{1}{2})$, $m\in[2,\infty)$ and let γ,p which satisfy (2.7). There exists a constant $C>0$ such that for any $0\leq S<T\leq 1$, any $\mathcal{F}_S$-measurable random variable η and any $(s,t)\in\Delta_{S,T}$, we have*

$$\Big\|\int_s^t b^n(\eta+K_r^{h,n}+B_r)-b^n(\eta+K_{r_h}^{h,n}+B_{r_h})\,dr\Big\|_{L^m}\leq C\Big(\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon}+\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\Big)(t-s)^{\frac{1}{2}}.$$

Proof. Define the process $\psi_t = \eta + K_t^{h,n}$, $t \in [0,1]$. Since ψ_t is $\mathcal{F}_t$-measurable for $t \in [S,T]$ and $b^n \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\gamma$, we can apply Corollary 5.5. It comes that

$$\begin{aligned}\Big\| \int_s^t & b^n(\eta + K_r^{h,n} + B_r) - b^n(\eta + K_{r_h}^{h,n} + B_{r_h})\, dr \Big\|_{L^m} \\ &\leq C\Big(\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{\varepsilon}{2}} + \|b^n\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[0,1]}L^m} h^{1-\varepsilon}(t-s)^{\frac{1}{2}} \Big)(t-s)^{\frac{1}{2}} \\ &\leq C\Big(\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[0,1]}L^m} h^{1-\varepsilon} \Big)(t-s)^{\frac{1}{2}}.\end{aligned}$$

It remains to prove an upper bound on $[\psi]_{\mathcal{C}^1_{[0,1]}L^m}$. For $0 \leq u \leq v \leq 1$, we have

$$|\psi_v - \psi_u| = |\int_u^v b^n(X_{r_h}^{h,n})\, dr| \leq |v-u| \|b^n\|_\infty.$$

Hence $[\psi]_{\mathcal{C}^1_{[0,1]}L^m} \leq \|b^n\|_\infty$. □

6 Examples and simulations

In this section, we discuss examples of SDEs of the form (1.1) that can be treated by Theorem 2.5.

6.1 Skew fractional Brownian motion

The skew Brownian motion is a one-dimensional process that behaves like a Brownian motion with a certain diffusion coefficient above the x-axis, and with another diffusion coefficient below the x-axis. We refer to [15, 22] for various constructions, and in particular in [21], it is shown to be the solution of an SDE which involves its local time. This equation reads $dX_t = \alpha\, dL_t^X + dW_t$, for $\alpha \in (-1,1)$, where L^X is the local time at 0 of the solution. Formally we can write $dL_t^X = \delta_0(X_t)\, dt$. More generally in $\mathbb{R}^d$, although this is not the only possible approach (see e.g. [4, 12] for alternative definitions), we call skew fractional Brownian motion the solution to (1.1) when the drift is $\alpha\delta_0$, $\alpha \in \mathbb{R}^d$, that is

$$dX_t = \alpha\delta_0(X_t)\, dt + dB_t. \tag{6.1}$$

Since $\delta_0 \in \mathcal{B}_p^{-d+\frac{d}{p}}$, Theorem 2.3 gives strong existence and uniqueness for $H < \frac{1}{2(d+1)}$ and the Euler scheme converges for the same values of H by Theorem 2.5.

As an alternative construction of the skew fBm, we also propose to replace the local time by its approximation $b(x) = \frac{\alpha}{2\varepsilon}\mathbb{1}_{(-\varepsilon,\varepsilon)}(x)$, $\varepsilon > 0$. Now we have b bounded and so $b \in \mathcal{B}_\infty^0$, therefore one can take $H < \frac{1}{2}$ and consider the SDE

$$dX_t = \frac{\alpha}{2\varepsilon}\mathbb{1}_{(-\varepsilon,\varepsilon)}(X)\, dt + dB_t.$$

In the Markovian case and dimension $d = 1$, the skew Brownian motion is reflected on the x-axis when $\alpha = \pm 1$. Unlike the skew Brownian motion, the skew fBm (for $H \neq 1/2$) is not reflected for any value of α, since $X - B$ is more regular than B (see Theorem 2.3). To construct reflected processes, a classical approach is to proceed by penalization, see e.g. [23] in the Brownian case, and [30] for rough differential equations. This consists in choosing a drift of the form $b_\varepsilon(x) = \frac{(x)_-}{\varepsilon}$ and letting ε tend to 0. Note that this approach also works for stochastic partial differential equations (SPDEs), see for instance [14, 27, 36]. If we consider more specifically the stochastic heat equation, the solution in time observed at a fixed point in space behaves qualitatively like a fractional SDE with Hurst parameter $H = \frac{1}{4}$. Hence in the reflected case, it is interesting to consider the following SDEs:

$$dX_t^\varepsilon = \frac{(X_t^\varepsilon)_-}{\varepsilon}\kappa(X_t^\varepsilon)\, dt + dB_t, \tag{6.2}$$

where κ is a smooth cut-off to ensure that the drift is in some Besov space (here $\mathcal{B}^1_\infty$). Therefore the analysis of the scheme (1.2) holds for $H < \frac{1}{2}$. One could then provide a rate for convergence for the Euler scheme approximating the reflected fBm X^0, which is the limit as $\varepsilon \to 0$ in (6.2). Indeed in [30], the distance between X^ε and X^0 is quantified, so denoting $X^{\varepsilon,h,n}$ the Euler scheme of X^ε, Theorem 2.5 now permits to control $X^0 - X^{\varepsilon,h,n}$.

6.2 Applications in finance

Some models of mathematical finance involve irregular drifts.

First, consider a dividend paying firm, whose capital evolution can be modelled by the following SDE:

$$dX_t = (r - \alpha(X_t))\,dt + \sigma\,dB_t,$$

with r is an interest rate, σ is the volatility of the market and $\alpha(x) = \mathbb{1}_{x\leq q}$ for some threshold q, see e.g [2]. Numerical methods for bounded drifts with Brownian noise exist in the literature, see e.g. [8, 17]. When B is a fractional Brownian motion with $H < 1/2$, [6] provides a rate of convergence for the strong error (and Theorem 2.5 provides the same rate of convergence).

Then, we propose a class of models which can be related heuristically to the rough Heston model introduced in [11]. Recently, it was observed empirically that the volatility in some high-frequency financial markets has a very rough behaviour, in the sense that its trajectories have a very small Hölder exponent, close to 0.1. Formally, the rough Heston model is described by a square root diffusion coefficient and a very rough driving noise. It would read

$$dV_t = \kappa(V_t)\,dt + \sqrt{V_t}\,dB_t, \tag{6.3}$$

if we could make sense of this equation, the difficulty being both to define a stochastic integral when H is small, and to ensure the positivity of the solution. Note that it is possible to define properly a rough Heston model, by means of Volterra equations, see [11]. However, we keep discussing (6.3) at a formal level, and consider the Lamperti transform $L(x) = \sqrt{x}$. Assume that a first order chain rule holds for the solution of (6.3), then as long as V stays nonnegative, it comes that

$$L(V_t) = L(V_0) + \int_0^t \frac{\kappa(V_s)}{\sqrt{V_s}}\,ds + B_t,$$

which for $\widetilde{V}_t := L(V_t) = \sqrt{V_t}$ also reads

$$\widetilde{V}_t = \widetilde{V}_0 + \frac{1}{2}\int_0^t \frac{1}{\widetilde{V}_s}\kappa(\widetilde{V}_s^2)\,ds + \frac{1}{2}B_t. \tag{6.4}$$

While there are some quantitative numerical approximation results for rough models (e.g. for the rough Bergomi model [13]), the Euler scheme for the rough Heston model is only known to converge without a rate [31]. Now we can make sense of the Equation (6.4) with drift $b(x) = \frac{\kappa(x^2)}{2|x|^{1-\varepsilon}}$ as for κ a bump function and for small $\varepsilon > 0$, $b \in \mathcal{B}^0_1$ (see [5, Prop. 2.21]). Hence Theorem 2.5 can be applied whenever $H < 1/4$ and in view of Corollary 2.6, this yields a strong error of order $(1/4)^-$.

Although the model (6.4) was not studied in the financial mathematics literature, it has the advantage of being rough, it is formally close to the rough Heston model and can be approximated quantitatively for $H < 1/4$.

6.3 Fractional Bessel processes in dimension 1

Bessel processes [29, Chapter XI] play an important role in probability theory and financial mathematics.

As a generalization and motivated by the discussion in the previous subsection, we consider solutions to the following one-dimensional SDE:

$$dX_t = \frac{\kappa(X_t)}{|X_t|^\alpha}\, dt + dB_t, \tag{6.5}$$

for some $\alpha > 0$ and $H \in (0,1)$. When $H = \frac{1}{2}$, $\alpha = 1$ and κ is the identity function, we know that the solution always stays positive [29, Chapter XI, Section 1]. By similar computations as in [5, Prop. 2.21], the drift $b(x) = \kappa(x)\,|x|^{-\alpha}$ belongs to $\mathcal{B}_\infty^{-\alpha}$ for $\alpha \in (0,1)$. In this case, assumption (2.7) reads $H < \frac{1}{2(1+\alpha)}$ and the rate of convergence of the Euler scheme is close to $\frac{1}{2(1+\alpha)}$.

6.4 Examples in higher dimension

A natural way to extend processes (6.5) to dimension 2 could be the following:

$$dX_t^i = \frac{\kappa(X_t)}{|X_t|^\alpha}\, dt + dB_t^i,\ i = 1, 2, \tag{6.6}$$

where B^1 and B^2 are two independent fBms and $\alpha > 0$. By [5, Proposition 2.21], one can prove that $x \mapsto b(x) = \frac{\kappa(x)}{|x|^\alpha}$ belongs to $\mathcal{B}_\infty^{-\alpha}$ for $\alpha \in (0,2)$. Therefore, the condition on H becomes $H < \frac{1}{2(1+\alpha)}$.

Notice that the SDE (6.6) presents a singularity only at the point $(0,0)$. To create a singularity on both the x and y-axes, one could also look at the following SDE

$$dX_t^i = \frac{1}{(|X_t^1| \wedge |X_t^2|)^\alpha}\, dt + dB_t^i,\ i = 1, 2.$$

Another example to consider in higher dimension is an SDE with discontinuous drift. For instance, let the drift be an indicator function of some domain D as in (6.7):

$$dX_t = \mathbb{1}_D^{(d)}(X_t)\, dt + dB_t, \tag{6.7}$$

where $\mathbb{1}_D^{(d)}$ denotes the vector-valued indicator function with identical entries $\mathbb{1}_D$ on each component. We have $\mathbb{1}_D^{(d)} \in \mathcal{B}_\infty^0$, and thus one can take $H < \frac{1}{2}$.

6.5 Simulations

In dimension 1, we will simulate two SDEs. First the skew fractional Brownian motion (6.1) with $\alpha = 1$. Then we simulate the SDE with bounded measurable drift $\mathbb{1}_{\mathbb{R}_+} \in \mathcal{B}_\infty^0$, i.e.

$$dX_t = \mathbb{1}_{X_t > 0}\, dt + dB_t. \tag{6.8}$$

The drifts are approximated by convolution with the Gaussian kernel, that is $b^n(x) = G_{\frac{1}{n}} b(x)$ and we fix the initial condition to $X_0 = 0$. For the skew fBm, this corresponds to

$$b^n(x) = \sqrt{\frac{n}{2\pi}} e^{-\frac{nx^2}{2}},$$

and for (6.8) this yields

$$b^n(x) = \sqrt{\frac{n}{2\pi}} \int_0^x e^{-\frac{ny^2}{2}}\, dy.$$

As in Corollary 2.6, we fix the time-step h of the Euler scheme as $h = \lfloor n^{-\frac{1}{1-\gamma+\frac{1}{p}}} \rfloor$. Our aim is to observe the rate of convergence numerically, so we need a reference value for the solutions of (6.1) and (6.8). However these solutions do not have an explicit expression so we do not have an exact reference value. Instead, we first make a costly computation with very small time-step $h = 10^{-6}$

that will serve as reference value. Therefore $n = h^{-2} = 1000$ for the skew fBm and $n = h^{-1}$ for the SDE with bounded drift. In a second step, we compute the Euler scheme for different values of h and compare it to the reference value with the same noise and $h = 10^{-6}$. The result is averaged over $N = 100$ realisations of the noise to get an estimate of the strong error.

Recall that according to Corollary 2.6, the theoretical order of convergence is almost $1/4$ when the drift is a Dirac and almost $1/2$ when the drift is bounded. We plot the logarithmic strong error with respect to the time-step h in Figure 1 and conclude that the empirical order of convergence is coherent with the theoretical one.

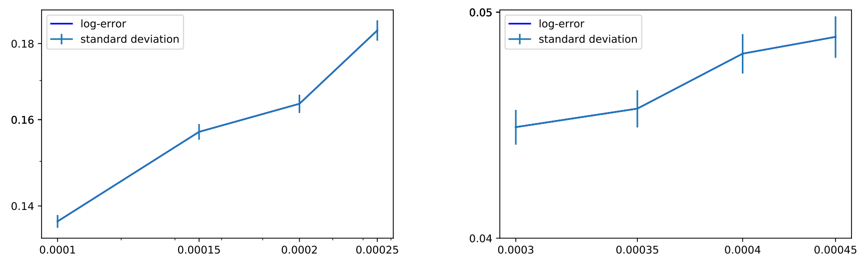

Figure 1: Plot of the logarithm of the strong error (y-axis) against h (x-axis) when $d = 1$. Left: the drift is the Dirac δ_0, $H = 0.2$, the numerical order of convergence (by linear regression) is approximately 0.27 ± 0.01, which is close to the theoretical order 0.25. Right: bounded drift, $H = 0.4$, the numerical order of convergence is approximately 0.55 ± 0.01. The theoretical order 0.5 is not in the interval 0.57 ± 0.01, which could be explained by other sources of numerical error (for instance the computation of the reference value).

In dimension 2, we simulate the 2-dimensional SDE (6.7) with $X_0 = 0$ and with D the quadrant defined by $D = \{x \geq 0, y \geq 0\}$. The drift b is approximated by

$$b^n(x) = G_{\frac{1}{n}} b(x) = \frac{n}{2\pi} \int_{\mathbb{R}^2} e^{-\frac{n}{2}|x-y|^2} \mathbb{1}_D(y)\, dy.$$

We recall that the time-step h is fixed such that $h = \lfloor n^{-\frac{1}{1-\gamma+\frac{2}{p}}} \rfloor$. In this case, $b \in \mathcal{B}^0_\infty$, thus we have $h = n^{-1}$. To obtain a reference value for the SDE, we set $h = 10^{-5}$. We compute the Euler scheme for a different values of h between 0.0003 and 0.0005 and compare it to the reference value with the same noise and $h = 10^{-5}$. The result is averaged over $N = 50$ realisations of the noise to get an estimate of the strong error.

Recall that according to Corollary 2.6, the theoretical order of convergence is almost $1/2$ when the drift is bounded. We plot the logarithmic strong error with respect to the time step h in Figure 2 and observe that the empirical order of convergence is approximately 0.62.

Appendices

A Proofs of regularisation by fBm in dimension d

We start by recalling the following recent stochastic sewing Lemma [20], which will be useful for the main estimate of this section (Proposition 3.5) and of Section 5.

Lemma A.1 ([20])**.** *Let $0 \leq S < T$ and $m \geq 2$. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Let $A : \Delta_{S,T} \to L^m$ be such that $A_{s,t}$ is $\mathcal{F}_t$-measurable for any $(s,t) \in \Delta_{S,T}$. Assume that there exist*

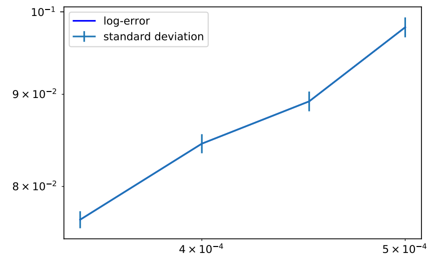

Figure 2: Plot of the logarithm of the strong error (y-axis) against h (x-axis) when $d = 2$. The drift is bounded, $H = 0.4$, the numerical order of convergence is approximately 0.62 ± 0.01, which is larger than the theoretical order. The theoretical order 0.5 is not in the interval 0.62 ± 0.01, which could be explained by other sources of numerical error (for instance the computation of the reference value).

constants $\Gamma_1, \Gamma_2 \geq 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that for every $(s,t) \in \Delta_{S,T}$ and $u := (s+t)/2$,

$$\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \leq \Gamma_1 \, (t-s)^{1+\varepsilon_1}, \tag{A.1}$$

$$\|\delta A_{s,u,t}\|_{L^m} \leq \Gamma_2 \, (t-s)^{\frac{1}{2}+\varepsilon_2}. \tag{A.2}$$

Then there exists a process $(\mathcal{A}_t)_{t \in [S,T]}$ such that, for any $t \in [S,T]$ and any sequence of partitions $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ of $[S,t]$ with mesh size going to zero, we have

$$\mathcal{A}_t = \lim_{k \to \infty} \sum_{i=0}^{N_k} A_{t_i^k, t_{i+1}^k} \text{ in probability.} \tag{A.3}$$

Moreover, there exists a constant $C = C(\varepsilon_1, \varepsilon_2, m)$ independent of S, T such that for every $(s,t) \in \Delta_{S,T}$ we have

$$\|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L^m} \leq C \, \Gamma_1 (t-s)^{1+\varepsilon_1} + C \, \Gamma_2 (t-s)^{\frac{1}{2}+\varepsilon_2},$$

and

$$\|\mathbb{E}^S[\mathcal{A}_t - \mathcal{A}_s - A_{s,t}]\|_{L^m} \leq C \, \Gamma_1 (t-s)^{1+\varepsilon_1}.$$

A.1 Proof of Lemma 3.4

We will apply Lemma A.1 for $S \leq s \leq t \leq T$,

$$\mathcal{A}_t := \int_S^t f(B_r, \Xi) \, dr \quad \text{and} \quad A_{s,t} := \mathbb{E}^s \left[\int_s^t f(B_r, \Xi) \, dr \right].$$

Notice that we have $\mathbb{E}^s[\delta A_{s,u,t}] = 0$, so (A.1) trivially holds. In order to establish (A.2), we will show that for some $\varepsilon_2 > 0$,

$$\|\delta A_{s,u,t}\|_{L^m} \leq \Gamma_2 \, (t-s)^{1/2+\varepsilon_2}. \tag{A.4}$$

For $u = (s+t)/2$ we have by the triangle inequality, Jensen's inequality for conditional expectation and Lemma 3.3(ii) that

$$\begin{aligned}
\|\delta A_{s,u,t}\|_{L^m} &\le \left\|\mathbb{E}^s\left[\int_u^t f(B_r,\Xi)\,dr\right]\right\|_{L^m} + \left\|\mathbb{E}^u\left[\int_u^t f(B_r,\Xi)\,dr\right]\right\|_{L^m} \\
&\le \int_u^t (\|\mathbb{E}^s f(B_r,\Xi)\|_{L^m} + \|\mathbb{E}^u f(B_r,\Xi)\|_{L^m})\,dr \\
&\le 2\int_u^t \|\mathbb{E}^u f(B_r,\Xi)\|_{L^m}\,dr \\
&\le C\int_u^t \|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\gamma}\|_{L^m}(r-u)^{H(\gamma-\frac{d}{p})}\,dr \\
&\le C\,\|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\gamma}\|_{L^m}\,(t-u)^{1+H(\gamma-\frac{d}{p})},
\end{aligned}$$

Hence, we have (A.4) for $\varepsilon_2 = 1/2 + H(\gamma - \frac{d}{p}) > 0$.

Let $t \in [S,T]$. Let $(\Pi_k)_{k\in\mathbb{N}}$ be a sequence of partitions of $[S,t]$ with mesh size converging to zero. For each k, denote $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$. By Lemma 3.3($iii$) we have that

$$\begin{aligned}
\|\mathcal{A}_t - \sum_i A_{t_i^k,t_{i+1}^k}\|_{L^1} &\le \sum_i \int_{t_i^k}^{t_{i+1}^k} \|f(B_r,\Xi) - \mathbb{E}^{t_i^k} f(B_r,\Xi)\|_{L^1}dr \\
&\le C\,\|\|f(\cdot,\Xi)\|_{\mathcal{C}^1}\|_{L^2}\,(t-S)\,|\Pi_k|^H \longrightarrow 0.
\end{aligned}$$

Hence (A.3) holds true.

Applying Lemma A.1, we get

$$\|\mathcal{A}_t - \mathcal{A}_S\|_{L^m} \le \|A_{S,t}\|_{L^m} + C\,\|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\gamma}\|_{L^m}\,(t-S)^{1+H(\gamma-\frac{d}{p})}.$$

Applying the triangle inequality and Lemma 3.3(ii), we get that

$$\begin{aligned}
\|A_{S,t}\|_{L^m} &= \left\|\mathbb{E}^S \int_S^t f(B_r,\Xi)\,dr\right\|_{L^m} \\
&\le \int_S^t \|\mathbb{E}^S f(B_r,\Xi)\|_{L^m}dr \\
&\le C\int_S^t \|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\gamma}\|_{L^m}\,(r-S)^{H(\gamma-\frac{d}{p})}dr \\
&\le C\,\|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\gamma}\|_{L^m}\,(t-S)^{1+H(\gamma-\frac{d}{p})},
\end{aligned}$$

and the result follows.

A.2 Proof of Proposition 3.5

Assume that $[\psi]_{\mathcal{C}^\alpha_{[s,t]}L^m} < \infty$, otherwise (3.2) trivially holds. For $(\tilde{s},\tilde{t}) \in \Delta_{s,t}$, let

$$A_{\tilde{s},\tilde{t}} := \int_{\tilde{s}}^{\tilde{t}} f(B_r + \psi_{\tilde{s}})dr \text{ and } \mathcal{A}_t := \int_s^{\tilde{t}} f(B_r + \psi_r)dr. \tag{A.5}$$

In the following, we check the conditions in order to apply Lemma A.1. To show that (A.1) and (A.2) hold true with $\varepsilon_1 = H(\gamma - d/p - 1) + \tau > 0$ and $\varepsilon_2 = 1/2 + H(\gamma - d/p) > 0$, we prove that there exists a constant $C > 0$ independent of $s, t, \tilde{s}$ and $\tilde{t}$ such that

(i) $\|\mathbb{E}^{\tilde{s}}[\delta A_{\tilde{s},u,\tilde{t}}]\|_{L^m} \le C\,\|f\|_{\mathcal{B}_p^\gamma}\,[\psi]_{\mathcal{C}^\tau[s,t]L^m}\,(\tilde{t}-\tilde{s})^{1+H(\gamma-1-d/p)+\tau}$;

(ii) $\|\delta A_{\tilde{s},u,\tilde{t}}\|_{L^m} \le C\,\|f\|_{\mathcal{B}_p^\gamma}\,(\tilde{t}-\tilde{s})^{1+H(\gamma-d/p)}$;

(iii) If (i) and (ii) are satisfied, (A.3) gives the convergence in probability of $\sum_{i=0}^{N_n-1} A_{t_i^k,t_{i+1}^k}$ along any sequence of partitions $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ of $[s,\tilde{t}]$ with mesh converging to 0. We will prove that the limit is the process $\mathcal{A}$ given in (A.5).

Assume for now that (i), (ii) and (iii) hold. Applying Lemma A.1, we obtain that

$$\begin{aligned}\Big\| \int_{\tilde{s}}^{\tilde{t}} f(B_r+\psi_r)\,dr\Big\|_{L^m} \leq & C\,\|f\|_{\mathcal{B}_p^\gamma}\,(\tilde{t}-\tilde{s})^{1+H(\gamma-\frac{d}{p})} \\ & + C\,\|f\|_{\mathcal{B}_p^\gamma}\,[\psi]_{\mathcal{C}^\tau[s,t]L^m}\,(\tilde{t}-\tilde{s})^{1+H(\gamma-1-\frac{d}{p})+\tau} \\ & + \|A_{\tilde{s},\tilde{t}}\|_{L^m}.\end{aligned}$$

We will see in (A.7) that $\|A_{\tilde{s},\tilde{t}}\|_{L^m} \leq C\,\|f\|_{\mathcal{B}_p^\gamma}\,(\tilde{t}-\tilde{s})^{1+H(\gamma-d/p)}$. Then, choosing $(\tilde{s},\tilde{t})=(s,t)$ we get (3.2).

We now check that the conditions (i), (ii) and (iii) actually hold.

Proof of (i): For $s\leq\tilde{s}\leq u\leq\tilde{t}\leq t$, we have

$$\delta A_{\tilde{s},u,\tilde{t}} = \int_u^{\tilde{t}} f(B_r+\psi_{\tilde{s}})-f(B_r+\psi_u)\,dr.$$

Hence, by the tower property of conditional expectation and Fubini's Theorem, we get

$$|\mathbb{E}^{\tilde{s}}\delta A_{\tilde{s},u,\tilde{t}}| = \Big|\mathbb{E}^{\tilde{s}}\int_u^{\tilde{t}} \mathbb{E}^u[f(B_r+\psi_{\tilde{s}})-f(B_r+\psi_u)]\,dr\Big|.$$

Now using Lemma 3.3(ii) with the $\mathcal{F}_u$-measurable variable $\Xi=(\psi_{\tilde{s}},\psi_u)$, Lemma 3.1$(ii)$ for $\alpha=1$ and again Fubini's Lemma, we obtain that

$$\begin{aligned}\Big|\mathbb{E}^{\tilde{s}}\int_u^{\tilde{t}} \mathbb{E}^u[f(B_r+\psi_{\tilde{s}})-f(B_r+\psi_u)]\,dr\Big| &\leq \mathbb{E}^{\tilde{s}}\int_u^{\tilde{t}} \|f(\cdot+\psi_{\tilde{s}})-f(\cdot+\psi_u)\|_{\mathcal{B}_p^{\gamma-1}}(r-u)^{H(\gamma-1-\frac{d}{p})}\,dr \\ &\leq C\|f\|_{\mathcal{B}_p^\gamma}\int_u^{\tilde{t}} \mathbb{E}^{\tilde{s}}[|\psi_u-\psi_{\tilde{s}}|](r-u)^{H(\gamma-1-\frac{d}{p})}\,dr.\end{aligned}$$

Hence we get

$$\|\mathbb{E}^{\tilde{s}}\delta A_{\tilde{s},u,\tilde{t}}\|_{L^m} \leq C\,\|f\|_{\mathcal{B}_p^\gamma}\int_u^{\tilde{t}} \|\mathbb{E}^{\tilde{s}}|\psi_u-\psi_{\tilde{s}}|\|_{L^m}(r-u)^{H(\gamma-1-\frac{d}{p})}dr. \tag{A.6}$$

By the conditional Jensen's inequality, we have

$$\|\mathbb{E}^{\tilde{s}}|\psi_u-\psi_{\tilde{s}}|\|_{L^m} \leq [\psi]_{\mathcal{C}^\tau_{[s,t]}L^m}(u-\tilde{s})^\tau.$$

Hence combining the previous inequality with equation (A.6), we get

$$\|\mathbb{E}^{\tilde{s}}\delta A_{\tilde{s},u,\tilde{t}}\|_{L^m} \leq C\,\|f\|_{\mathcal{B}_p^\gamma}\,[\psi]_{\mathcal{C}^\tau_{[s,t]}L^m}(\tilde{t}-\tilde{s})^{1+H(\gamma-1-\frac{d}{p})+\tau}.$$

Proof of (ii): This is a direct consequence of Lemma 3.4. Indeed, we write

$$\|\delta A_{\tilde{s},u,\tilde{t}}\|_{L^m} \leq \|A_{\tilde{s},\tilde{t}}\|_{L^m}+\|A_{\tilde{s},u}\|_{L^m}+\|A_{u,\tilde{t}}\|_{L^m}$$

and we will apply Lemma 3.4 for each term in the right-hand side of the previous inequality, respectively for $\Xi=\psi_{\tilde{s}}$, $\psi_{\tilde{s}}$ again and ψ_u. We only detail the first one. As f is smooth and bounded, the first assumption of Lemma 3.4 is verified. By Lemma 3.1(i), $\|f(\cdot+\Xi)\|_{\mathcal{B}_p^\gamma} \leq \|f\|_{\mathcal{B}_p^\gamma}$, hence the second assumption of Lemma 3.4 is verified. It follows that

$$\begin{aligned}\|A_{\tilde{s},\tilde{t}}\|_{L^m} &\leq C\,\|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\gamma}\|_{L^m}\,(\tilde{t}-\tilde{s})^{1+H(\gamma-\frac{d}{p})} \\ &\leq C\,\|f\|_{\mathcal{B}_p^\gamma}\,(\tilde{t}-\tilde{s})^{1+H(\gamma-\frac{d}{p})}.\end{aligned} \tag{A.7}$$

Hence combining similar inequalities on $A_{\tilde{s},u}$ and $A_{u,\tilde{t}}$ with (A.7), we get

$$\|\delta A_{\tilde{s},u,\tilde{t}}\|_{L^m} \leq C\,\|f\|_{\mathcal{B}_p^\gamma}\,(\tilde{t}-\tilde{s})^{1+H(\gamma-\frac{d}{p})}.$$

Proof of (iii): For a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[s,\tilde{t}]$ with $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$ and mesh size converging to zero, we have

$$\begin{aligned}\Big\|\mathcal{A}_{\tilde{t}} - \sum_i A_{t_i^k,t_{i+1}^k}\Big\|_{L^m} &\leq \sum_i \int_{t_i^k}^{t_{i+1}^k} \|f(B_r+\psi_r) - f(B_r+\psi_{t_i^k})\|_{L^m}\,dr \\ &\leq \sum_i \int_{t_i^k}^{t_{i+1}^k} \|f\|_{\mathcal{C}^1}\|\psi_r - \psi_{t_i^k}\|_{L^m}\,dr \\ &\leq C\,\|f\|_{\mathcal{C}^1}\,t\,|\Pi_k|^\tau\,[\psi]_{\mathcal{C}^\tau_{[s,t]}L^m} \underset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

A.3 Proof of Proposition 3.9

This proof is very close to the proof of [1, Proposition 7.7], but we adapt it to dimension $d \geq 1$ for the reader's convenience.

Assume w.l.o.g. that $X_0 = 0$ and let $\tilde{K} := \tilde{X} - \tilde{B}$, so that (2.4) is automatically verified. Let $(b^n)_{n\in\mathbb{N}}$ be any sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\gamma-}$. To verify that $\tilde{K}$ and $\tilde{X}$ satisfy (2.5), we have to show that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left|\int_0^t b^n(\tilde{X}_r)\,dr - \tilde{K}_t\right| = 0 \text{ in probability.} \tag{A.8}$$

By the triangle inequality we have that for $k,n\in\mathbb{N}$ and $t\in[0,T]$,

$$\begin{aligned}\left|\int_0^t b^n(\tilde{X}_r)\,dr - \tilde{K}_t\right| &\leq \left|\int_0^t b^n(\tilde{X}_r)\,dr - \int_0^t b^n(\tilde{X}_r^k)\,dr\right| + \left|\int_0^t b^n(\tilde{X}_r^k)\,dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k)\,dr\right| \\ &\quad + \left|\int_0^t \tilde{b}^k(\tilde{X}_r^k)\,dr - \tilde{K}_t\right| =: A_1 + A_2 + A_3.\end{aligned} \tag{A.9}$$

Now we will show that all summands on the right hand side of (A.9) converge to 0 uniformly on $[0,T]$ in probability as $n\to\infty$, choosing $k = k(n)$ accordingly.

First we bound A_1. Notice that

$$\begin{aligned}\left|\int_0^t b^n(\tilde{X}_r)\,dr - \int_0^t b^n(\tilde{X}_r^k)\,dr\right| &\leq \|b^n\|_{\mathcal{C}^1}\int_0^t |\tilde{X}_r - \tilde{X}_r^k|\,dr \\ &\leq \|b^n\|_{\mathcal{C}^1}\,T\sup_{t\in[0,T]}|\tilde{X}_t - \tilde{X}_t^k|.\end{aligned}$$

For any $\varepsilon > 0$, choose an increasing sequence $(k(n))_{n\in\mathbb{N}}$ such that

$$\mathbb{P}\Big(\|b^n\|_{\mathcal{C}^1}\,T\sup_{t\in[0,T]}|\tilde{X}_t - \tilde{X}_t^{k(n)}| > \varepsilon\Big) < \frac{1}{n},\ \forall n\in\mathbb{N}.$$

Hence, we get that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left|\int_0^t b^n(\tilde{X}_r)\,dr - \int_0^t b^n(\tilde{X}_r^{k(n)})\,dr\right| = 0 \text{ in probability.}$$

Now, we bound A_2. Let $\gamma' < \gamma$ with $\gamma' - d/p > 1/2 - 1/(2H)$. By Lemma 3.7 applied to $\tilde{X}^k$, $h = b^n - \tilde{b}^k$ and γ' instead of γ, there exists a random variable $Z_{n,k}$ such that

$$\begin{aligned}\mathbb{E}[Z_{n,k}] &\leq C\,\|b^n - \tilde{b}^k\|_{\mathcal{B}_p^{\gamma'}}\,(1 + \|\tilde{b}^k\|^2_{\mathcal{B}_p^{\gamma'}}) \\ &\leq C\,(\|b^n - b\|_{\mathcal{B}_p^{\gamma'}} + \|\tilde{b}^k - b\|_{\mathcal{B}_p^{\gamma'}})\,(1 + \sup_{m\in\mathbb{N}}\|\tilde{b}^m\|^2_{\mathcal{B}_p^{\gamma'}}),\end{aligned} \tag{A.10}$$

for C independent of k, n and such that there is

$$\sup_{t\in[0,T]} \left| \int_0^t b^n(\tilde{X}_r^k)\, dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k)\, dr \right| \leq Z_{n,k}(1+T).$$

Using Markov's inequality and (A.10) we obtain that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left| \int_0^t b^n(\tilde{X}_r^k)\, dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k)\, dr \right| > \varepsilon \right) \leq \varepsilon^{-1}\, \mathbb{E}[Z_{n,k}]\,(1+T)$$
$$\leq C\, \varepsilon^{-1}\,(1+T)\,(\|b^n - b\|_{\mathcal{B}_p^{\gamma'}} + \|\tilde{b}^k - b\|_{\mathcal{B}_p^{\gamma'}})\,(1 + \sup_{m\in\mathbb{N}} \|\tilde{b}^m\|^2_{\mathcal{B}_p^{\gamma'}}).$$

Choosing $k = k(n)$ as before, we get

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t b^n(\tilde{X}_r^{k(n)})\, dr - \int_0^t \tilde{b}^{k(n)}(\tilde{X}_r^{k(n)})\, dr \right| = 0 \text{ in probability.}$$

To bound the last summand A_3, recall that $\tilde{X}_t^k = \int_0^t \tilde{b}^k(\tilde{X}_r^k)\, dr + \tilde{B}_t^k$. We get that

$$\sup_{t\in[0,T]} \left| \int_0^t \tilde{b}^k(\tilde{X}_r^k)\, dr - \tilde{K}_t \right| \leq \sup_{t\in[0,T]} (|\tilde{X}_t^k - \tilde{X}_t| + |\tilde{B}_t^k - \tilde{B}_t|).$$

Since by assumption $(\tilde{X}^k, \tilde{B}^k)_{k\in\mathbb{N}}$ converges to $(\tilde{X}, \tilde{B})$ on $(\mathcal{C}_{[0,T]})^2$ in probability, we get that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \tilde{b}^{k(n)}(\tilde{X}_r^{k(n)})\, dr - \tilde{K}_t \right| = 0 \text{ in probability,}$$

and therefore (A.8) holds true.

It remains to show that (3.6) holds true. By Lemma 3.6, there exists $C > 0$ such that for any $(s,t) \in \Delta_{0,T}$,

$$\|(\tilde{X}_t^k - \tilde{B}_t^k) - (\tilde{X}_s^k - \tilde{B}_s^k)\|_{L^m} \leq C\,(1 + \sup_{m\in\mathbb{N}} \|\tilde{b}^m\|^2_{\mathcal{B}_p^{\gamma}})\,(t-s)^{1+H(\gamma - \frac{d}{p})}. \tag{A.11}$$

Using that $\int_0^t \tilde{b}^k(\tilde{X}_r^k)\, dr$ converges to K_t on $\mathcal{C}_{[0,T]}$ in probability (by assumption) and that $\sup_{m\in\mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_p^{\gamma}}$ is finite, we get (3.6) by applying Fatou's Lemma to (A.11).

References

[1] L. Anzeletti, A. Richard, and E. Tanré. Regularisation by fractional noise for one-dimensional differential equations with nonnegative distributional drift. *Preprint arXiv:2112.05685*, 2021.

[2] S. Asmussen and M. Taksar. Controlled diffusion models for optimal dividend pay-out. *Insurance Math. Econom.*, 20(1):1–15, 1997.

[3] S. Athreya, O. Butkovsky, K. Lê, and L. Mytnik. Well-posedness of stochastic heat equation with distributional drift and skew stochastic heat equation. *Preprint arXiv:2011.13498*, 2020.

[4] D. Baños, S. Ortiz-Latorre, A. Pilipenko, and F. Proske. Strong solutions of stochastic differential equations with generalized drift and multidimensional fractional Brownian initial noise. *J. Theoret. Probab.*, 35(2):714–771, 2022.

[5] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343. Springer, 2011.

[6] O. Butkovsky, K. Dareiotis, and M. Gerencsér. Approximation of SDEs: a stochastic sewing approach. *Probab. Theory Related Fields*, 181(4):975–1034, 2021.

[7] R. Catellier and M. Gubinelli. Averaging along irregular curves and regularisation of ODEs. *Stochastic Process. Appl.*, 126(8):2323–2366, 2016.

[8] K. Dareiotis, M. Gerencsér, and K. Lê. Quantifying a convergence theorem of Gyöngy and Krylov. *Preprint arXiv:2101.12185*, 2021.

[9] A. M. Davie. Uniqueness of solutions of stochastic differential equations. *Int. Math. Res. Not. IMRN*, (24):Art. ID rnm124, 26, 2007.

[10] T. De Angelis, M. Germain, and E. Issoglio. A numerical scheme for stochastic differential equations with distributional drift. *Preprint arXiv:1906.11026*, 2019.

[11] O. El Euch and M. Rosenbaum. The characteristic function of rough Heston models. *Math. Finance*, 29(1):3–38, 2019.

[12] J. Garzón, J. A. León, and S. Torres. Fractional stochastic differential equation with discontinuous diffusion. *Stoch. Anal. Appl.*, 35(6):1113–1123, 2017.

[13] P. Gassiat. Weak error rates of numerical schemes for rough volatility. *Preprint arXiv:2203.09298*, 2022.

[14] L. Goudenège. Stochastic Cahn-Hilliard equation with singular nonlinearity and reflection. *Stochastic Process. Appl.*, 119(10):3516–3548, 2009.

[15] J. M. Harrison and L. A. Shepp. On skew Brownian motion. *Ann. Probab.*, 9(2):309–313, 1981.

[16] Y. Hu, Y. Liu, and D. Nualart. Rate of convergence and asymptotic error distribution of Euler approximation schemes for fractional diffusions. *Ann. Appl. Probab.*, 26(2):1147–1207, 2016.

[17] B. Jourdain and S. Menozzi. Convergence rate of the Euler-Maruyama scheme applied to diffusion processes with $L^Q - L^\rho$ drift coefficient and additive noise. *Preprint arXiv:2105.04860*, 2021.

[18] P. E. Kloeden and E. Platen. Stochastic differential equations. In *Numerical Solution of Stochastic Differential Equations*, pages 103–160. Springer, 1992.

[19] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005.

[20] K. Lê. A stochastic sewing lemma and applications. *Electron. J. Probab.*, 25:Paper No. 38, 55, 2020.

[21] J.-F. Le Gall. One-dimensional stochastic differential equations involving the local times of the unknown process. In *Stochastic analysis and applications (Swansea, 1983)*, volume 1095 of *Lecture Notes in Math.*, pages 51–82. Springer, Berlin, 1984.

[22] A. Lejay. On the constructions of the skew Brownian motion. *Probab. Surv.*, 3:413–466, 2006.

[23] P.-L. Lions and A.-S. Sznitman. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.*, 37(4):511–537, 1984.

[24] J.-C. Mourrat and H. Weber. Global well-posedness of the dynamic Φ^4 model in the plane. *Ann. Probab.*, 45(4):2398–2476, 2017.

[25] A. Neuenkirch and I. Nourdin. Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion. *J. Theoret. Probab.*, 20(4):871–899, 2007.

[26] D. Nualart and Y. Ouknine. Regularization of differential equations by fractional noise. *Stochastic Process. Appl.*, 102(1):103–116, 2002.

[27] D. Nualart and E. Pardoux. White noise driven quasilinear SPDEs with reflection. *Probab. Theory Related Fields*, 93(1):77–89, 1992.

[28] E. Pardoux and D. Talay. Discretization and simulation of stochastic differential equations. *Acta Appl. Math.*, 3(1):23–47, 1985.

[29] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Fundamental Principles of Mathematical Sciences*. Springer-Verlag, Berlin, third edition, 1999.

[30] A. Richard, E. Tanré, and S. Torres. Penalisation techniques for one-dimensional reflected rough differential equations. *Bernoulli*, 26(4):2949–2986, 2020.

[31] A. Richard, X. Tan, and F. Yang. On the discrete-time simulation of the rough Heston model. *Preprint arXiv:2107.07835*, 2021.

[32] F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields*, 97(3):403–421, 1993.

[33] M. Szölgyenyi. Stochastic differential equations with irregular coefficients: mind the gap! *Preprint arXiv:2104.11505*, 2021.

[34] D. Talay and L. Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic Anal. Appl.*, 8(4):483–509 (1991), 1990.

[35] A. J. Veretennikov. Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb. (N.S.)*, 111(153)(3):434–452, 480, 1980.

[36] L. Zambotti. Integration by parts on δ-Bessel bridges, $\delta > 3$ and related SPDEs. *Ann. Probab.*, 31(1):323–348, 2003.

[37] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974.