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**Abstract**

We study the well-posedness and numerical approximation of multidimensional stochastic differential equations (SDEs) with distributional drift, driven by a fractional Brownian motion.

First, we prove weak existence for such SDEs. This holds under a condition that relates the Hurst parameter $H$ of the noise to the Besov regularity of the drift. Then under a stronger condition, we study the error between a solution $X$ of the SDE with drift $b$ and its tamed Euler scheme with mollified drift $b^n$. We obtain a rate of convergence in $L^m(\Omega)$ for this error, which depends on the Besov regularity of the drift. This result covers the critical case of the regime of strong existence and pathwise uniqueness. When the Besov regularity increases and the drift becomes a bounded measurable function, we recover the (almost) optimal rate of convergence $1/2-\varepsilon$.

As a byproduct of this convergence, we deduce that pathwise uniqueness holds in a class of Hölder continuous solutions and that any such solution is strong. The proofs rely on stochastic sewing techniques, especially to deduce new regularising properties of the discrete-time fractional Brownian motion. We also present several examples and numerical simulations that illustrate our results.

***Keywords and phrases:*** *Numerical approximation, regularisation by noise, fractional Brownian motion.*

**MSC2020 subject classification:** 60H10, 65C30, 60G22, 60H50, 34A06.

## 1 Introduction

We are interested in the well-posedness and numerical approximation of the following $d$-dimensional SDE:

$$X_t = X_0 + \int_0^t b(X_s)\,ds + B_t, \quad t \in [0,1], \tag{1.1}$$

where $X_0 \in \mathbb{R}^d$, $b$ is a distribution in some nonhomogeneous Besov space $\mathcal{B}_p^\gamma$ and $B$ is an $\mathbb{R}^d$-fractional Brownian motion (fBm) with Hurst parameter $H$. When $B$ is a standard Brownian motion ($H=1/2$), this equation received a lot of attention when the drift is irregular, see for instance [41, 44] for bounded measurable drift or [23] under some integrability condition. Strong well-posedness was obtained in those cases, which contrasts with the non-uniqueness and sometimes non-existence that can happen for the corresponding equations without noise. In case $B$ is a fractional Brownian motion, the results are more recent and we refer to Nualart and Ouknine [31]

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for Hölder continuous drifts, then to Baños et al. [4], Catellier and Gubinelli [7], Galeati et al. [15], Anzeletti et al. [1] and Galeati and Gerencsér [14] for distributional drifts when the Hurst parameter is smaller than $1/2$.

The most simple approximation scheme for (1.1) is given by the Euler scheme with a time-step $h$

$$X_t^h = X_0 + \int_0^t b(X_{r_h}^h)\, dr + B_t, \quad t \in [0,1],$$

where $r_h = h\lfloor \frac{r}{h} \rfloor$. For the numerical analysis of Brownian SDEs with smooth coefficients, including the previous scheme and higher-order approximations, we point to a few classical works by Pardoux, Talay and Tubaro [33, 40], see also [22]. The strong error $\|X_t - X_t^h\|_{L^m(\Omega)}$ is known to be of order $h$ (and $h^{1/2}$ when the noise is multiplicative). When the coefficients are irregular, Dareiotis et al. [8] obtained recently a strong error with the optimal rate of order $1/2$ for merely bounded measurable drifts, even if the noise is multiplicative. This was extended to integrable drifts with a Krylov-Röckner condition by Lê and Ling [25]. We also refer to the review [39] and references therein for discontinuous coefficients, and the recent weak error analysis of Jourdain and Menozzi [21] for integrable drifts. Besides, we mention that when the drift is a distribution in a Bessel potential space with negative regularity, De Angelis et al. [10] have obtained a rate of convergence for the so-called virtual solutions of a (Brownian) SDE, using a 2-step mollification procedure of the drift.

Let us now recall briefly what is known when $B$ is a fractional Brownian motion. First, Neuenkirch and Nourdin [30] considered one-dimensional equations with $H > 1/2$, smooth coefficients and multiplicative noise, i.e. the more general case with $B$ replaced by a symmetric Russo-Vallois [38] integral $\int_0^t \sigma(X_s)\, d^o B_s$ in (1.1). They proved that the rate of convergence for the strong error is exactly of order $2H - 1$. Then Hu et al. [20] introduced a modified Euler scheme to obtain an improved convergence rate of order $2H - 1/2$, still in the multiplicative case. They also derived an interesting weak error rate of convergence. Recently, Butkovsky et al. [6] considered (1.1) with any Hurst parameter $H \in (0,1)$ and Hölder continuous drifts in $\mathcal{C}^\alpha$, for $\alpha \in [0,1]$. They obtained the strong error convergence rate $h^{(1/2+\alpha H)\wedge 1-\varepsilon}$, which holds whenever $\alpha \geq 0$ and $\alpha > 1 - 1/(2H)$. The latter condition is optimal in the sense that it corresponds to the existence and uniqueness result for (1.1) established in [7]. Our main contribution in this paper is an extension of their result to distributional drifts, i.e. to negative values of $\alpha$, including the threshold $\alpha = 1 - 1/(2H)$.

First, we state that if $b$ is in the Besov space $\mathcal{B}_p^\gamma$ and that $\gamma - d/p > 1/2 - 1/(2H)$, then there exists a weak solution $(X, B)$ to (1.1) which has some Hölder regularity. This result is a direct extension of [1, Theorem 2.8] to any dimension $d \geq 1$ and was also recently extended to time-dependent drifts in [14]. The condition $\gamma - d/p > 1/2 - 1/(2H)$ allows negative values of $\gamma$ and therefore $b$ can be a genuine distribution. Solutions to (1.1) are then understood as processes of the form $X_t = X_0 + K_t + B_t$, where $K_t$ is the limit of $\int_0^t b^n(X_s)\, ds$, for any approximating sequence $(b^n)_{n\in\mathbb{N}}$. We see in particular that this approach is well suited for numerical approximation. Hence we propose a numerical scheme to approximate (1.1).

To that end, for a time-step $h$ and a sequence $(b^n)_{n\in\mathbb{N}}$ that converges to $b$ in a Besov sense, we consider the following tamed Euler scheme that we define on the same probability space and with the same fBm $B$ as $X$:

$$X_t^{h,n} = X_0 + \int_0^t b^n(X_{r_h}^{h,n})\, dr + B_t, \tag{1.2}$$

where $r_h = h\lfloor \frac{r}{h} \rfloor$. Choosing $b^n = g_{1/n} * b$ as a convolution of $b$ with the Gaussian density $g_{1/n}$ with variance $1/n$, and for a careful choice of $n$ as a function of $h$, we prove under the stronger condition $\gamma - d/p > 1 - 1/(2H)$ that the following rate of convergence holds

$$\forall h \in (0,1), \quad \sup_{t\in[0,1]} \|X_t - X_t^{h,n}\|_{L^m(\Omega)} \leq C h^{\frac{1}{2(1-\gamma+\frac{d}{p})}-\varepsilon}.$$

A more general version of this result is presented in Theorem 2.5 and discussed thereafter, in particular concerning the value of the rate. We also obtain a non-explicit rate of convergence in the limit case of the strong regime, that is when $\gamma - d/p = 1 - 1/(2H)$ and $\gamma > 1 - 1/(2H)$.

This extends the result of Butkovsky et al. [6] to negative values of $\alpha \equiv \gamma - d/p$, and matches the $1/2 - \varepsilon$ rate of convergence obtained in the limit case $\gamma - d/p = 0$ and $b$ a bounded measurable function. Unlike previous works, our method does not rely on the Girsanov transform and avoids computing exponential moments of functionals of the noise or its discrete-time approximation.

As a byproduct, we deduce that under the condition $\gamma - d/p \geq 1 - 1/(2H)$ and $\gamma > 1 - 1/(2H)$, $X$ is in fact a strong solution and it is pathwise unique in a class of Hölder continuous processes. Note that in the sub-critical case $\gamma - d/p > 1 - 1/(2H)$, a notion of uniqueness (path-by-path) was already proven in [7], and strong existence was established in [15], for solutions in the sense of nonlinear Young differential equations. We compare these results to ours in Remark 2.4.

Our proof relies on several new regularisation properties of the $d$-dimensional fBm and of the discrete-time fBm which somehow extend Davie's lemma [9, Prop. 2.1]. Namely, for functions $f$ in Besov spaces of negative regularity (resp. bounded $f$ for the discrete-time fBm), we obtain upper bounds on the moments of quantities such as $\int_s^t f(x + B_r)\, dr$ in terms of $x$ and $(t - s)$, see Propositions 3.5, 5.2 and 5.11. These upper bounds are sharper than if $B$ was replaced by any smooth function, which explains that we refer to regularisation properties of the fBm. The main tool to prove these results is the stochastic sewing lemma developed by Lê [24].

The critical case with $\gamma - d/p = 1 - 1/(2H)$ and $\gamma > 1 - 1/(2H)$ requires a version of the stochastic sewing lemma with critical exponents that induces a logarithmic factor in the result (see [3, Theorem 4.5] and [12, Lemma 4.10]). We use this lemma in Proposition 5.4 to prove an upper bound on the moments of $\int_s^t f(X_r) - f(X_r^{h,n})\, dr$. This leads to the following bound for $\mathcal{E}^{h,n} = X - X^{h,n}$,

$$\begin{aligned}\|\mathcal{E}_t^{h,n} - \mathcal{E}_s^{h,n}\|_{L^m} \leq C \left(\|\mathcal{E}^{h,n}\|_{L^\infty_{[s,t]} L^m} + \epsilon(h,n)\right) (t-s)^{\frac{1}{2}} \\ + C \left(\|\mathcal{E}^{h,n}\|_{L^\infty_{[s,t]} L^m} + \epsilon(h,n)\right) \left|\log\left(\|\mathcal{E}^{h,n}\|_{L^\infty_{[s,t]} L^m} + \epsilon(h,n)\right)\right| (t-s),\end{aligned}$$

for some $\epsilon(h,n) = o(1)$. We then prove a Grönwall-type lemma with logarithmic factors (Lemma 4.1) which yields a control of $\|\mathcal{E}^{h,n}\|_{L^\infty_{[0,1]} L^m}$ by a power of $\epsilon(h,n)$.

**Organisation of the paper.** We start with definitions and notations in Subsection 2.1, then state our main results in Subsection 2.2. In Section 3, we first recall some Besov estimates in Subsection 3.1, then state a first regularisation property of the fBm in Subsection 3.2. We prove tightness and stability results in Subsection 3.3. We conclude the proof of weak existence and use the convergence of the tamed Euler scheme to establish strong existence and uniqueness in Subsection 3.4. The strong convergence of the numerical scheme (1.2) to the solution of (1.1) is established in Section 4. This proof relies strongly on the regularisation lemmas for fBm and discrete-time fBm which are stated and proven in Section 5. In Section 6, we provide examples of SDEs that can be approximated with our result. We also run simulations of the scheme (1.2) to study its empirical rate of convergence and compare it with the theoretical result. Finally we gather some technical proofs based on the stochastic sewing lemma in Appendix A and complete the proof of uniqueness in a larger class of processes in Appendix B.

# 2 Framework and results

## 2.1 Notations and definitions

In this section, we define notations that are used throughout the paper.

- On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ a filtration that satisfies the usual conditions.

- The conditional expectation given $\mathcal{F}_t$ is denoted by $\mathbb{E}^t$ when there is no risk of confusion on the underlying filtration.
- An $\mathbb{R}^d$-valued stochastic process $(X_t)_{t\in[0,1]}$ is said to be adapted if for all $t \in [0,1]$, $X_t$ is $\mathcal{F}_t$-measurable.
- For $\alpha \in [0,1]$, $I$ a subset of $[0,1]$ and $E$ a Banach space, we denote by $\mathcal{C}^\alpha_I E$ the space of $E$-valued mappings that are $\alpha$-Hölder continuous on $I$. The corresponding semi-norm for a function $f : [0,1] \to E$ reads

$$[f]_{\mathcal{C}^\alpha_I E} := \sup_{\substack{s,t\in I\\ t\neq s}} \frac{\|f_t - f_s\|_E}{|t-s|^\alpha}.$$

- The $L^m(\Omega)$ norm, $m \in [1,\infty]$, of a random variable $X$ is denoted by $\|X\|_{L^m}$ and the space $L^m(\Omega)$ is simply denoted by $L^m$. In that case we denote by $\mathcal{C}^\alpha_I L^m$ the space of $L^m(\Omega)$-valued mappings that are $\alpha$-Hölder continuous on $I$. For an $\mathbb{R}^d$-valued process $Z$, the corresponding semi-norm is then denoted by $[Z]_{\mathcal{C}^\alpha_I L^m}$.
- When $E$ is $\mathbb{R}$ or $\mathbb{R}^d$, we simply denote by $\mathcal{C}^\alpha_I$ the corresponding space and when $\alpha = 0$, we use the notation $\mathcal{C}_I$.
- We write $L^\infty_I$ for the space of bounded measurable functions on a subset $I$ of $[0,1]$ and $L^\infty_I L^m := L^\infty(I, L^m(\Omega))$. For a Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}^d$, denote the classical $L^\infty$ and $\mathcal{C}^1$ norms of $f$ by $\|f\|_\infty = \sup_{x\in\mathbb{R}^d}|f(x)|$ and $\|f\|_{\mathcal{C}^1} = \|f\|_\infty + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}$.
The corresponding norm for a process $Z : [0,1]\times\Omega \to \mathbb{R}^d$ is

$$\|Z\|_{L^\infty_I L^m} := \sup_{s\in I}\|Z_s\|_{L^m}.$$

- For all $S,T \in [0,1]$, define the simplex $\Delta_{S,T}$ by

$$\Delta_{S,T} = \{(s,t) \in [S,T], s < t\}.$$

- For a process $Z : \Delta_{0,1} \times \Omega \to \mathbb{R}^d$, we still write

$$[Z]_{\mathcal{C}^\alpha_I L^m} = \sup_{\substack{s,t\in I\\ s<t}} \frac{\|Z_{s,t}\|_{L^m}}{|t-s|^\alpha} \quad \text{and} \quad \|Z\|_{L^\infty_I L^m} = \sup_{s\in I}\|Z_{0,s}\|_{L^m}.$$

- In applications of the stochastic sewing lemma, we will need to consider increments of $Z$, which are given for any triplet of times $(s,u,t)$ such that $s \le u \le t$ by

$$\delta Z_{s,u,t} := Z_{s,t} - Z_{s,u} - Z_{u,t}.$$

- Finally, given a process $Z : [0,1]\times\Omega \to \mathbb{R}^d$, $\alpha \in (0,1]$, $m \in [1,\infty)$ and $q \in [1,\infty]$, we consider the following seminorm: for any $0 \le s \le t \le 1$,

$$[Z]_{\mathcal{C}^\alpha_{[s,t]} L^{m,q}} := \sup_{(u,v)\in\Delta_{s,t}} \frac{\|\mathbb{E}^u[|Z_v - Z_u|^m]^{\frac{1}{m}}\|_{L^q}}{(v-u)^\alpha}, \tag{2.1}$$

where the conditional expectation is taken with respect to the filtration the space is equipped with. By the tower property and Jensen's inequality for conditional expectation, we know that

$$[Z]_{\mathcal{C}^\alpha_{[s,t]} L^m} = [Z]_{\mathcal{C}^\alpha_{[s,t]} L^{m,m}} \le [Z]_{\mathcal{C}^\alpha_{[s,t]} L^{m,\infty}}. \tag{2.2}$$

**Heat kernel.** For any $t > 0$, denote by $g_t$ the Gaussian kernel on $\mathbb{R}^d$ with variance $t$:

$$g_t(x) = \frac{1}{(2\pi\, t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right),$$

and by $G_t$ the associated Gaussian semigroup on $\mathbb{R}^d$: for $f : \mathbb{R}^d \to \mathbb{R}^d$,

$$G_t f(x) = \int_{\mathbb{R}^d} g_t(x-y)\, f(y)\, dy. \tag{2.3}$$

**Besov spaces.** We use the same definition of nonhomogeneous Besov spaces as in [5], which we write here for any dimension $d$. Let $\chi, \varphi : \mathbb{R}^d \to \mathbb{R}$ be the smooth radial functions which are given by [5, Proposition 2.10], with $\chi$ supported on a ball while $\varphi$ is supported on an annulus. Let $v_{-1}$ and $v$ respectively be the inverse Fourier transform of $\chi$ and $\varphi$. Denote by $\mathcal{F}$ the Fourier transform and $\mathcal{F}^{-1}$ its inverse.

The nonhomogeneous dyadic blocks $\Delta_j$, $j \in \mathbb{N} \cup \{-1\}$ are defined for any $\mathbb{R}^d$-valued tempered distribution $u$ by

$$\Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u) \quad \text{and} \quad \Delta_j u = \mathcal{F}^{-1}\left(\varphi(2^{-j}\cdot)\mathcal{F} u\right) \quad \text{for } j \geq 0.$$

Let $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$. We denote by $\mathcal{B}_p^\gamma$ the nonhomogeneous Besov space $\mathcal{B}_{p,\infty}^\gamma(\mathbb{R}^d, \mathbb{R}^d)$ of $\mathbb{R}^d$-valued tempered distributions $f$ such that

$$\|f\|_{\mathcal{B}_p^\gamma} = \sup_{j \geq -1} 2^{j\gamma} \|\Delta_j f\|_{L_p(\mathbb{R}^d)} < \infty.$$

Let $1 \leq p_1 \leq p_2 \leq \infty$. The space $\mathcal{B}_{p_1}^\gamma$ continuously embeds into $\mathcal{B}_{p_2}^{\gamma - d(1/p_1 - 1/p_2)}$, which we write as $\mathcal{B}_{p_1}^\gamma \hookrightarrow \mathcal{B}_{p_2}^{\gamma - d(1/p_1 - 1/p_2)}$, see e.g. [5, Prop. 2.71].

Finally, we denote by $C$ a constant that can change from line to line and that does not depend on any parameter other than those specified in the associated lemma, proposition or theorem. When we want to make the dependence of $C$ on some parameter $a$ explicit, we will write $C(a)$.

To give a meaning to equation (1.1) with distributional drift, we first need to precise in which sense those drifts are approximated.

**Definition 2.1.** *Let $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$. We say that a sequence of smooth bounded functions $(b^n)_{n \in \mathbb{N}}$ converges to $b$ in $\mathcal{B}_p^{\gamma-}$ as $n$ goes to infinity if*

$$\begin{cases} \sup_{n \in \mathbb{N}} \|b^n\|_{\mathcal{B}_p^\gamma} \leq \|b\|_{B_p^\gamma} < \infty, \\ \lim_{n \to \infty} \|b^n - b\|_{\mathcal{B}_p^{\gamma'}} = 0, \quad \forall \gamma' < \gamma. \end{cases} \tag{2.4}$$

Following [31], in dimension $d = 1$, we recall a notion of $\mathbb{F}$-fBm which extends the classical definition of $\mathbb{F}$-Brownian motion. There exists a one-to-one operator $\mathcal{A}_H$ (which can be written explicitly in terms of fractional derivatives and integrals, see [1, Definition 2.3]) such that for $B$ an fBm, the process $W := \mathcal{A}_H B$ is a Brownian motion. Then we say that $B$ is an $\mathbb{F}$-fBm if $W$ is an $\mathbb{F}$-Brownian motion. In any dimension $d \geq 1$, we say that $B$ is an $\mathbb{R}^d$-valued $\mathbb{F}$-fBm, if each component is an $\mathbb{F}$-fBm.

We are now ready to introduce the notions of solution to (1.1).

**Definition 2.2.** *Let $\gamma \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\gamma$, $T > 0$ and $X_0 \in \mathbb{R}^d$. As in [1], we define the following notions.*

- Weak solution: *a couple $((X_t)_{t \in [0,1]}, (B_t)_{t \in [0,1]})$ defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a weak solution to (1.1) on $[0, 1]$, with initial condition $X_0$, if*
  - *$B$ is an $\mathbb{R}^d$-valued $\mathbb{F}$-fBm;*
  - *$X$ is adapted to $\mathbb{F}$;*
  - *there exists an $\mathbb{R}^d$-valued process $(K_t)_{t \in [0,1]}$ such that, a.s.,*

$$X_t = X_0 + K_t + B_t \text{ for all } t \in [0, 1]; \tag{2.5}$$

  - *for every sequence $(b^n)_{n \in \mathbb{N}}$ of smooth bounded functions converging to $b$ in $\mathcal{B}_p^{\gamma-}$, we have that*

$$\sup_{t \in [0,1]} \left| \int_0^t b^n(X_r) dr - K_t \right| \underset{n \to \infty}{\longrightarrow} 0 \text{ in probability.} \tag{2.6}$$

*If the couple is clear from the context, we simply say that* $(X_t)_{t\in[0,1]}$ *is a weak solution.*

- Pathwise uniqueness: *As in the classical literature on SDEs, we say that pathwise uniqueness holds if for any two solutions* $(X,B)$ *and* $(Y,B)$ *defined on the same filtered probability space with the same fBm* $B$ *and same initial condition* $X_0 \in \mathbb{R}^d$, $X$ *and* $Y$ *are indistinguishable.*

- Strong solution: *A weak solution* $(X,B)$ *such that* $X$ *is* $\mathbb{F}^B$*-adapted is called a strong solution, where* $\mathbb{F}^B$ *denotes the filtration generated by* $B$.

## 2.2 Main results

Our first result is decomposed into two parts: first, it gives a condition for existence of a weak solution to (1.1) and therefore extends [1, Theorem 2.8] to the multidimensional setting. The proof is presented in Section 3. The second part gives existence and uniqueness of a strong solution under stronger assumptions, and will be a consequence of the convergence of the tamed Euler scheme in Theorem 2.5. Thus it provides a multidimensional extension of [1, Theorem 2.9] through a completely different proof.

**Theorem 2.3.** *Let* $\gamma \in \mathbb{R}$, $p \in [1,\infty]$ *and* $b \in \mathcal{B}_p^\gamma$.

*(a) Assume that*

$$0 > \gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}. \tag{H1}$$

*Then there exists a weak solution* $X$ *to* (1.1) *such that* $[X-B]_{\mathcal{C}^\kappa_{[0,1]}L^{m,\infty}} < \infty$ *for any* $\kappa \in (0, 1 + H(\gamma - d/p) \wedge 0] \setminus \{1\}$ *and* $m \geq 2$.

*(b) Assume that*

$$H < \frac{1}{2}, \;\; 0 > \gamma - \frac{d}{p} \geq 1 - \frac{1}{2H} \;\; \text{and} \;\; \gamma > 1 - \frac{1}{2H}. \tag{H2}$$

*Then there exists a strong solution* $X$ *to* (1.1) *such that* $[X-B]_{\mathcal{C}^{1/2+H}_{[0,1]}L^{m,\infty}} < \infty$ *for any* $m \geq 2$. *Besides, pathwise uniqueness holds in the class of all solutions* $X$ *such that* $[X-B]_{\mathcal{C}^{1/2+H}_{[0,1]}L^{2,\infty}} < \infty$. *Finally, if* $\gamma - d/p > 1 - 1/(2H)$, *pathwise uniqueness holds in the class of all solutions* $X$ *such that* $[X-B]_{\mathcal{C}^{H(1-\gamma+d/p)+\eta}_{[0,1]}L^{2,\infty}} < \infty$, *for any* $\eta \in (0,1)$.

The proof of Theorem 2.3($a$) is given in Section 3.4 and the proof of Theorem 2.3($b$) in Section 3.5. The latter follows from the convergence of the tamed Euler scheme stated in Corollary 2.6: since the scheme is adapted to $\mathbb{F}^B$ and converges to any weak solution $X$ such that $[X-B]_{\mathcal{C}^{1/2+H}_{[0,1]}L^{2,\infty}} < \infty$, we deduce uniqueness and also that the weak solution is adapted to $\mathbb{F}^B$ and is therefore a strong one. For $\eta \in (0,1)$, we extend the uniqueness result to solutions $X$ that satisfy $[X-B]_{\mathcal{C}^{H(1-\gamma+d/p)+\eta}_{[0,1]}L^{2,\infty}} < \infty$ in Appendix B.

**Remark 2.4.**

- *In [7] and more recently [14, 15],* (1.1) *was solved in the sense of nonlinear Young differential equations in the sub-critical case of the strong regime: for* $b$ *in the Hölder space* $\mathcal{C}^\alpha$ *(=* $\mathcal{B}^\alpha_\infty$ *when* $\alpha \notin \mathbb{N}$*), strong existence and path-by-path uniqueness hold for* $\alpha > 1 - 1/(2H)$*, unconditionally. Nevertheless the notions of solution might not be equivalent: we know from Theorem 2.14(a) in [1] that a nonlinear Young solution with some Hölder regularity (which is proven to hold in [15]) is a solution in the sense of Definition 2.2; however a strong solution in the sense of Definition 2.2 must also have some Hölder regularity to be a nonlinear Young solution (Theorem 2.14(b) in [1]) and we do not know if any strong solution has such regularity.*

- *In the weak regime $\gamma - d/p > 1/2 - 1/(2H)$, weak existence was proven in [1] in dimension 1, then extended in higher dimension with time-dependence in [14]. In [14], the notion of solution is similar to Definition 2.2, with $X$ that must satisfy* (2.5) *and* (2.6) *for any approximating sequence $(b_n)$ that converges to $b$ in $\mathcal{B}_p^\gamma$, which is slightly more restrictive than what is asked here (convergence in $\mathcal{B}_p^{\gamma-}$). Hence for the sake of completeness, we provide a proof of Theorem 2.3(a).*

- *Although the uniqueness result holds only in a class of regular enough processes, we will see in the next theorem that the Euler scheme chooses exactly the unique solution in this class.*

Let $(b^n)_{n\in\mathbb{N}}$ be a sequence of smooth functions that converges to $b$ in $\mathcal{B}_p^{\gamma-}$. Consider the tamed Euler scheme (1.2) associated to (1.1) with a time-step $h \in (0,1)$. The main result of this paper is the following theorem. It describes the convergence of the tamed Euler scheme to a weak solution $X$ such that $X - B \in \mathcal{C}^{1/2+H}_{[0,1]} L^{2,\infty}$. Choosing appropriately $h$ and $n$, we also deduce that the regularity of the tamed Euler scheme is the same, that is, $X^{h,n} - B \in \mathcal{C}^{1/2+H}_{[0,1]} L^{2,\infty}$.

**Theorem 2.5.** *Let $H < 1/2$, $\gamma \in \mathbb{R}$, $p \in [1,\infty]$ satisfying* (H2) *and let $m \in [2,\infty)$. Let $b \in \mathcal{B}_p^\gamma$ and $(b^n)_{n\in\mathbb{N}}$ be a sequence of smooth functions that converges to $b$ in $\mathcal{B}_p^{\gamma-}$. Let $X_0$ be an $\mathcal{F}_0$-measurable random variable, $(X,B)$ be a weak solution to* (1.1) *and $(X^{h,n})_{h\in(0,1),n\in\mathbb{N}}$ be the tamed Euler scheme defined in* (1.2)*, on the same probability space and with the same fBm $B$ as $X$.*

*(a) Regularity of the tamed Euler scheme: Let $\eta \in (0,H)$, $\mathcal{D}$ a sub-domain of $(0,1)\times\mathbb{N}$ and assume that*

$$\sup_{(h,n)\in\mathcal{D}} \|b^n\|_\infty h^{\frac{1}{2}-H} < \infty \quad \text{and} \quad \sup_{(h,n)\in\mathcal{D}} \|b^n\|_{\mathcal{C}^1} h^{\frac{1}{2}+H-\eta} < \infty. \tag{H3}$$

*Then* $\sup_{(h,n)\in\mathcal{D}} [X^{h,n} - B]_{\mathcal{C}^{\frac{1}{2}+H}_{[0,1]} L^{m,\infty}} < \infty.$

*Assume that $X - B \in \mathcal{C}^{1/2+H}_{[0,1]} L^{m,\infty}$ and let $\varepsilon \in (0,1/2)$.*

*(b) The sub-critical case: Assume $\gamma - d/p \in (1 - 1/(2H), 0)$. Then there exists $C > 0$ that depends on $m, p, \gamma, d, \varepsilon, \|b\|_{\mathcal{B}_p^\gamma}$ such that for all $h \in (0,1)$ and $n \in \mathbb{N}$, the following bound holds:*

$$[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C \left( \|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} + \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_\infty \|b^n\|_{\mathcal{C}^1} h^{1-\varepsilon} \right). \tag{2.7}$$

*(c) The critical case: Assume $\gamma - d/p = 1 - 1/(2H)$ and $\gamma > 1 - 1/(2H)$. Let $\zeta \in (0,1/2)$, $\mathbf{M}$ be the constant given by Proposition 5.4, and $\delta \in (0, e^{-\mathbf{M}})$. If* (H3) *holds, then there exists $C > 0$ that depends on $m, p, \gamma, d, \varepsilon, \zeta, \delta, \|b\|_{\mathcal{B}_p^\gamma}$ such that for all $(h,n) \in \mathcal{D}$, the following bound holds:*

$$\begin{aligned}[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[0,1]} L^m} \leq C \Big( &\|b - b^n\|_{\mathcal{B}_p^{\gamma-1}} (1 + |\log(\|b - b^n\|_{\mathcal{B}_p^{\gamma-1}})|) + \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} \\ &+ \|b^n\|_{\mathcal{C}^1} \|b^n\|_\infty h^{1-\varepsilon} \Big)^{(e^{-\mathbf{M}}-\delta)}.\end{aligned} \tag{2.8}$$

Obviously, the previous error bounds also hold for the strong error in uniform norm, since we have

$$\sup_{t\in[0,1]} \|X_t - X_t^{h,n}\|_{L^m} \leq [X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[0,1]} L^m}. \tag{2.9}$$

Although the Hurst parameter does not appear in the upper bounds (2.7)-(2.8), the first term $\|b^n - b\|_{\mathcal{B}_p^{\gamma-1}}$ does depend implicitly on $H$ through (H2). Observe also that the second term, $\|b^n\|_\infty h^{1/2-\varepsilon}$, corresponds to the optimal rate of convergence found in [6].

In the upper bounds (2.7)-(2.8), it is important to choose carefully the sequence $(b^n)_{n\in\mathbb{N}}$ to obtain a good rate of convergence of the numerical scheme. Choosing $b^n = G_{1/n}$ for $n \in \mathbb{N}^*$, we have thanks to Lemma 3.2 that for $\gamma - d/p < 0$,

$$\|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} \leq C\,\|b\|_{\mathcal{B}_p^\gamma}\; n^{-\frac{1}{2}}, \tag{2.10}$$

$$\|b^n\|_\infty \leq C\,\|b\|_{\mathcal{B}_p^\gamma}\; n^{-\frac{1}{2}(\gamma-\frac{d}{p})}, \tag{2.11}$$

$$\|b^n\|_{\mathcal{C}^1} \leq C\,\|b\|_{\mathcal{B}_p^\gamma}\, n^{\frac{1}{2}}\; n^{-\frac{1}{2}(\gamma-\frac{d}{p})}. \tag{2.12}$$

Using these results in (2.7) and optimising over $n$ and $h$, we deduce the following corollary.

**Corollary 2.6.** *Let the same assumptions as in Theorem 2.5 hold. For $h \in (0, 1/2)$, define*

$$n_h = \left\lfloor h^{-\frac{1}{1-\gamma+\frac{d}{p}}} \right\rfloor \quad \textit{and} \quad b^{n_h} = G_{\frac{1}{n_h}} b.$$

*Then we have*

$$\sup_{h\in(0,\frac{1}{2})} [X^{h,n_h} - B]_{\mathcal{C}^{\frac{1}{2}+H}_{[0,1]} L^{m,\infty}} < \infty. \tag{2.13}$$

*Let $\varepsilon \in (0, 1/2)$.*

*(a)* <u>*The sub-critical case*</u>*: Assume $\gamma - d/p \in (1 - 1/(2H), 0)$. Then there exists $C > 0$ that depends on $m, p, \gamma, \varepsilon, \|b\|_{\mathcal{B}_p^\gamma}$ such that the following bound holds:*

$$\forall h \in \left(0, \frac{1}{2}\right), \quad [X - X^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C\, h^{\frac{1}{2(1-\gamma+\frac{d}{p})}-\varepsilon}. \tag{2.14}$$

*(b)* <u>*The critical case*</u>*: Assume $\gamma - d/p = 1 - 1/(2H)$ and $\gamma > 1 - 1/(2H)$. Let $\zeta \in (0, 1/2)$, $\mathbf{M}$ be the constant given by Proposition 5.4, and $\delta \in (0, e^{-\mathbf{M}})$. Then there exists $C > 0$ that depends on $m, p, \gamma, \varepsilon, \zeta, \delta, \|b\|_{\mathcal{B}_p^\gamma}$ such that the following bound holds:*

$$\forall h \in \left(0, \frac{1}{2}\right), \quad [X - X^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[0,1]} L^m} \leq C\, h^{H(e^{-\mathbf{M}}-\delta)}. \tag{2.15}$$

**Remark 2.7.** *We construct the tamed Euler scheme on a particular probability space that is given in an abstract way by Theorem 2.3. From Corollary 2.6, we deduce (see Subsection 3.5) that $X$ is in fact a strong solution. It is then possible to construct the tamed Euler scheme on any probability space (rich enough to contain an $\mathbb{F}$-fBm), which is of practical importance for simulations.*

**Remark 2.8.** *For instance, if each component of $b$ is a signed measure, then $b \in \mathcal{B}_1^0 := \mathcal{B}_1^0(\mathbb{R}^d, \mathbb{R}^d)$ (see [5, Proposition 2.39]). Hence the previous result (Corollary 2.6(a)) yields a rate $\frac{1}{2(1+d)} - \varepsilon$, which holds for $H < \frac{1}{2(1+d)}$. In the critical case, when $H = \frac{1}{2(1+d)}$, the rate becomes $He^{-\mathbf{M}} - \varepsilon$.*

For $\gamma - d/p > 0$, $\mathcal{B}_p^\gamma$ is continuously embedded in the Hölder space $\mathcal{C}^{\gamma-d/p}$. In [6], it was proved that the Euler scheme achieves a rate $1/2 + H(\gamma - d/p) - \varepsilon$. Moreover, if $b$ is a bounded measurable function, the rate is $1/2 - \varepsilon$. To close the gap between the present results and [6], we handle the case $\gamma - d/p = 0$. Recall that $\mathcal{B}_p^{d/p}$ is continuously embedded into $\mathcal{B}_\infty^0$, so it is equivalent to work with $\gamma = 0$ and $p = +\infty$. Note that $\mathcal{B}_\infty^0$ contains strictly $L^\infty(\mathbb{R}^d)$ (see e.g. [37, Section 2.2.2, eq (8) and Section 2.2.4, eq (4)]) which was the space considered in [6]. Let $b \in \mathcal{B}_\infty^0$. By the definition of Besov spaces, we know that $b \in \mathcal{B}_\infty^{-\eta}$ for all $\eta > 0$. Choosing $\eta$ small enough so that $-\eta > 1 - 1/(2H)$, we can apply Theorem 2.3 and Theorem 2.5, and obtain a rate of convergence as in Corollary 2.6 when $b^n = G_{\frac{1}{n}} b$. This is summarized in the following Corollary.

**Corollary 2.9.** *Let the assumptions of Theorem 2.5 hold. Let $B$ be an $\mathbb{F}$-fBm with $H < 1/2$, $b \in \mathcal{B}_\infty^0$ and $m \geq 2$. There exists a strong solution $X$ to (1.1) such that $[X - B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{m,\infty}} < \infty$. Besides, for any $\eta > 0$, pathwise uniqueness holds in the class of solutions $X$ such that $[X - B]_{\mathcal{C}^{H+\eta}_{[0,1]} L^{2,\infty}} < \infty$.*

*Let $\varepsilon \in (0, 1/2)$. Then Theorem 2.5(a) holds and there exists a constant $C$ that depends only on $m, \varepsilon, \|b\|_{\mathcal{B}_\infty^0}$ such that for any $h \in (0, 1/2)$ and $n \in \mathbb{N}$, the following bound holds:*

$$[X - X^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C \left( \|b^n - b\|_{\mathcal{B}_\infty^{-1}} + \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_\infty \|b^n\|_{\mathcal{C}^1} h^{1-\varepsilon} \right).$$

*Moreover, for $n_h = \lfloor h^{-1} \rfloor$ and $b^{n_h} = G_{\frac{1}{n_h}} b$, we have*

$$\forall h \in \left(0, \frac{1}{2}\right), \quad [X - X^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C h^{\frac{1}{2}-\varepsilon},$$
$$\sup_{h \in (0, \frac{1}{2})} [X^{h,n_h} - B]_{\mathcal{C}^{\frac{1}{2}+H}_{[0,1]} L^{m,\infty}} < \infty.$$

Theorem 2.5, Corollary 2.6 and Corollary 2.9 are proven in Section 4.

## 2.3 Discussion on the approach and results

The main novelty of the paper is that we treat fractional SDEs with distributional drifts, including the critical case $\gamma - d/p = 1 - 1/(2H)$ and $\gamma > 1 - 1/(2H)$. For bounded drifts, one can use Girsanov's theorem (see e.g. [6, Lemma 4.2]), which requires upper bounds on exponential moments of functionals of the fBm. In the literature, this usually leads to an exponential dependence on $\|b^n\|_\infty$, which we chose to avoid. We note however that Lê and Ling [25] managed to develop a Girsanov argument for unbounded drifts for the Brownian motion, using the $L_t^q L^p(\mathbb{R}^d)$ norm of $b^n$ on small time intervals (see [25, Lemma 5.14]). For the fBm, the computations for a Girsanov argument do not seem to work with the stochastic sewing, since the functionals that appear include the fractional kernel (see for example [6, (B.1)]). As a novel approach, we use the stochastic sewing lemma to regularise directly integrals of functions of the discrete noise $\{X^{h,n}_{t_h}\}_{t \geq 0}$ (see Proposition 5.11) to avoid a Girsanov argument and any exponential dependence on norms of $b^n$. The price to pay is that the $\mathcal{C}^1$ norm of $b^n$ appears, which can be compensated by powers of $h$.

Let us make a few comments on the rate of convergence obtained in Corollaries 2.6 and 2.9:

- For fixed $\gamma, p$ and $d$, one can chose $H$ close to $\frac{1}{2(1-\gamma+\frac{d}{p})}$ from below, and get an order of convergence that will be $\frac{1}{2(1-\gamma+\frac{d}{p})} - \varepsilon \approx H - \varepsilon$.
- For a fixed $H$, one can take $b \in \mathcal{B}_\infty^{1-\frac{1}{2H}+\varepsilon}$ for any $\varepsilon > 0$, and get an order of convergence that will be close to $H$.
- The order of convergence is $1/2 - \varepsilon$ when $\gamma - d/p = 0$, for any $H < 1/2$.
- The order of convergence is $He^{-\mathbf{M}} - \varepsilon$ for some constant $\mathbf{M}$, if $\gamma - d/p = 1 - 1/(2H)$ and $\gamma > 1 - 1/(2H)$. Hence the rates of convergence we obtained change abruptly when $\gamma - d/p$ reaches the threshold $1 - 1/(2H)$.

In view of [6], one could have guessed that the order of convergence for $\gamma - d/p \leq 0$ would still be $1/2 + H(\gamma - d/p)$. In particular, we point out that the two orders, $1/2 + H(\gamma - d/p)$ and $\frac{1}{2(1-\gamma+d/p)}$, coincide when $\gamma - d/p = 1 - 1/(2H)$. However (H2) only implies the following inequality

$$\frac{1}{2} + H\left(\gamma - \frac{d}{p}\right) \geq \frac{1}{2(1-\gamma+\frac{d}{p})}.$$

Finally, the orders of convergence obtained here and in [6] are summarized in Table 1.

| | | | | |
|---|---|---|---|---|
| *Drift* | $\gamma-\frac{d}{p}=1-\frac{1}{2H}$ and $\gamma>1-\frac{1}{2H}$ | $\gamma-\frac{d}{p}\in(1-\frac{1}{2H},0)$ | $\gamma-\frac{d}{p}=0$ | $\gamma-\frac{d}{p}>0$ |
| *Rate* | $He^{-\mathbf{M}}-\varepsilon$ | $\frac{1}{2(1-\gamma+\frac{d}{p})}-\varepsilon$ | $\frac{1}{2}-\varepsilon$ | $\left(\frac{1}{2}+H(\gamma-\frac{d}{p})\right)\wedge 1-\varepsilon$ |

Table 1: Rate of convergence of the tamed Euler scheme depending on the Besov regularity of the drift.

# 3 Existence and uniqueness of solutions

In this section, we prove Theorem 2.3. The proof of Theorem 2.3($a$) follows the same lines as the proof of Theorem 2.8 of [1], but requires extensions of some technical lemmas concerning the regularising effects of the fractional Brownian motion in dimension $d$. Subsection 3.5 is dedicated to the proof of Theorem 2.3($b$).

## 3.1 Besov estimates

The first of these extensions concern estimates of shift of distributions in Besov spaces. It is an generalisation to $\mathbb{R}^d$ of Lemma A.2 in [3] and follows its proof exactly, so we omit it.

**Lemma 3.1.** *Let $f$ be a tempered distribution on $\mathbb{R}^d$ and let $\beta\in\mathbb{R}$, $p\in[1,\infty]$. Then for any $a_1,a_2,a_3\in\mathbb{R}^d$ and $\alpha,\alpha_1,\alpha_2\in[0,1]$, one has*

*(i)* $\|f(a+\cdot)\|_{\mathcal{B}^\beta_p}\leq\|f\|_{\mathcal{B}^\beta_p}$ .

*(ii)* $\|f(a_1+\cdot)-f(a_2+\cdot)\|_{\mathcal{B}^\beta_p}\leq C|a_1-a_2|^\alpha\|f\|_{\mathcal{B}^{\beta+\alpha}_p}$ .

*(iii)* $\|f(a_1+\cdot)-f(a_2+\cdot)-f(a_3+\cdot)+f(a_3+a_2-a_1+\cdot)\|_{\mathcal{B}^\beta_p}\leq C|a_1-a_2|^{\alpha_1}|a_1-a_3|^{\alpha_2}\|f\|_{\mathcal{B}^{\beta+\alpha_1+\alpha_2}_p}$.

Then we have the following estimates for the Gaussian semigroup in Besov spaces. They are either borrowed or adapted from [3, 5].

**Lemma 3.2.** *Let $\beta\in\mathbb{R}$, $p\in[1,\infty]$ and $f\in\mathcal{B}^\beta_p$. Then*

*(i) If $\beta<0$, $\|G_tf\|_{L^p(\mathbb{R}^d)}\leq C\,\|f\|_{\mathcal{B}^\beta_p}\,t^{\frac{\beta}{2}}$, for all $t>0$.*

*(ii) If $\beta-\frac{d}{p}<0$, $\|G_tf\|_\infty\leq C\,\|f\|_{\mathcal{B}^\beta_p}\,t^{\frac{1}{2}(\beta-\frac{d}{p})}$, for all $t>0$.*

*(iii) $\|G_tf-f\|_{\mathcal{B}^{\beta-\varepsilon}_p}\leq C\,t^{\frac{\varepsilon}{2}}\,\|f\|_{\mathcal{B}^\beta_p}$ for all $\varepsilon\in(0,1]$ and $t>0$. In particular, it follows that $\lim_{t\to0}\|G_tf-f\|_{\mathcal{B}^{\tilde\beta}_p}=0$ for every $\tilde\beta<\beta$.*

*(iv) $\sup_{t>0}\|G_tf\|_{\mathcal{B}^\beta_p}\leq\|f\|_{\mathcal{B}^\beta_p}$.*

*(v) If $\beta-\frac{d}{p}<0$, $\|G_tf\|_{\mathcal{C}^1}\leq C\,\|f\|_{\mathcal{B}^\beta_p}\,t^{\frac{1}{2}(\beta-\frac{d}{p}-1)}$ for all $t>0$.*

*Proof.* *(i)* The proof of Lemma A.3($i$) in [3] extends right away to dimension $d\geq1$.

*(ii)* Using ($i$) for $\beta-\frac{d}{p}$ instead of $\beta$ and the embedding $\mathcal{B}^\beta_p\hookrightarrow\mathcal{B}^{\beta-\frac{d}{p}}_\infty$, there is

$$\|G_tf\|_{L^\infty(\mathbb{R}^d)}\leq C\,\|f\|_{\mathcal{B}^{\beta-d/p}_\infty}\,t^{\frac{1}{2}(\beta-\frac{d}{p})}\leq C\,\|f\|_{\mathcal{B}^\beta_p}\,t^{\frac{1}{2}(\beta-\frac{d}{p})}.$$

*(iii)* This is an adaptation of Lemma A.3($ii$) in [3] to dimension $d\geq1$ that we detail briefly. From [29, Lemma 4], we have that for $g$ such that the support of $\mathcal{F}g$ is in a ball of radius $\lambda\geq1$, there is for all $t\geq0$,

$$\|G_tg-g\|_{L^p(\mathbb{R}^d)}\leq C\,(t\lambda^2\wedge1)\|g\|_{L^p(\mathbb{R}^d)}.$$

For any $j \geq -1$, the support of $\mathcal{F}(\Delta_j f)$ is included in a ball of radius $2^j$. Hence,

$$\begin{aligned} 2^{j(\beta-\varepsilon)} \|G_t(\Delta_j f) - \Delta_j f\|_{L^p(\mathbb{R}^d)} &\leq C\, 2^{j(\beta-\varepsilon)} (t 2^{2j} \wedge 1) \|\Delta_j f\|_{L^p(\mathbb{R}^d)} \\ &\leq C\, 2^{-j\varepsilon} (t 2^{2j} \wedge 1)^{\frac{\varepsilon}{2}}\, 2^{j\beta} \|\Delta_j f\|_{L^p(\mathbb{R}^d)} \\ &\leq C\, t^{\frac{\varepsilon}{2}}\, 2^{j\beta} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

The result follows.

*(iv)* – *(v)* The proof is the same as in the one-dimensional case, see [3, Lemma A.3(*iii*)] and [3, Lemma A.3(*iv*)].

□

The next lemma describes some time regularity estimates of random functions of the fractional Brownian motion in Besov norms.

**Lemma 3.3.** *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and $B$ be an $\mathbb{F}$-fBm. Let $\beta < 0$, $p \in [1, \infty]$ and $e \in \mathbb{N}^*$. Then there exists a constant $C > 0$ such that for any $(s,t) \in \Delta_{0,1}$, any bounded measurable function $f : \mathbb{R}^d \times \mathbb{R}^e \to \mathbb{R}^d$ and any $\mathcal{F}_s$-measurable $\mathbb{R}^e$-valued random variable $\Xi$ satisfying $\|f(\cdot, \Xi)\|_{\mathcal{C}^1} < \infty$ almost surely, there is*

*(i)* $\mathbb{E}^s[f(B_t, \Xi)] = G_{\sigma^2_{s,t}} f(\mathbb{E}^s[B_t], \Xi)$, *where $G$ is the Gaussian semigroup introduced in* (2.3) *and* $\sigma^2_{s,t} := \mathrm{Var}\,(B^{(i)}_t - \mathbb{E}^s[B^{(i)}_t])$, *for any component $B^{(i)}$ of the fBm;*

*(ii)* $|\mathbb{E}^s f(B_t, \Xi)| \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}^\beta_p} (t-s)^{H(\beta - \frac{d}{p})}$;

*(iii)* $\|f(B_t, \Xi) - \mathbb{E}^s f(B_t, \Xi)\|_{L^1} \leq C \|\|f(\cdot, \Xi)\|_{\mathcal{C}^1}\|_{L^2} (t-s)^H$ .

*(iv)* *Furthermore, for any $u$ in the interval $(s,t)$ and $m \in [1,p]$ there exists a constant $C > 0$ such that* $\|\mathbb{E}^u \left[ f\left(B_t, \Xi\right)\right]\|_{L^m} \leq C \left\| \|f(\cdot, \Xi)\|_{\mathcal{B}^\beta_p} \right\|_{L^m} (t-u)^{H\beta} (u-s)^{-\frac{d}{2p}} (t-s)^{d\frac{1-2H}{2p}}$.

*Proof.* The proofs of $(i)$, $(ii)$, $(iii)$ are similar to $(a)$, $(b)$, $(c)$ in [1, Lemma 5.1] but they rely now on Lemma 3.1 and Lemma 3.2. We only reproduce the proof of $(iv)$ which is similar to $(d)$ in [1, Lemma 5.1], to emphasize where the dimension $d$ appears.

For $u \in (s,t)$, we have from $(i)$ that

$$\begin{aligned} \mathbb{E}^s |\mathbb{E}^u [f(B_t, \Xi)]|^m &= \mathbb{E}^s |G_{\sigma^2_{u,t}} f(\mathbb{E}^u B_t, \Xi)|^m \\ &= \mathbb{E}^s |G_{\sigma^2_{u,t}} f(\mathbb{E}^u B_t - \mathbb{E}^s B_t + \mathbb{E}^s B_t, \Xi)|^m. \end{aligned}$$

Notice that $\mathbb{E}^s B_t$ is independent of $\mathbb{E}^u B_t - \mathbb{E}^s B_t$, which is a Gaussian variable with mean zero and covariance $\sigma^2_{s,u,t} I_d$ where

$$\sigma^2_{s,u,t} := \mathrm{Var}(\mathbb{E}^u B^{(i)}_t - \mathbb{E}^s B^{(i)}_t),$$

for any component $B^{(i)}$ of the fBm. It follows that

$$\mathbb{E}^s |\mathbb{E}^u [f(B_t, \Xi)]|^m = \int_{\mathbb{R}^d} g_{\sigma^2_{s,u,t}}(y) |G_{\sigma^2_{u,t}} f(\mathbb{E}^s B_t + y, \Xi)|^m \, dy.$$

Let $q = \frac{p}{m}$ and $q' = \frac{q}{q-1}$. Using Hölder's inequality, we get

$$\begin{aligned} \mathbb{E}^s |\mathbb{E}^u [f(B_t, \Xi)]|^m &\leq \|g_{\sigma^2_{s,u,t}}\|_{L^{q'}(\mathbb{R}^d)} \|G_{\sigma^2_{u,t}} f(\cdot, \Xi)\|^m_{L^p(\mathbb{R}^d)} \\ &= \|G_{\sigma^2_{s,u,t}} \delta_0\|_{L^{q'}(\mathbb{R}^d)} \|G_{\sigma^2_{u,t}} f(\cdot, \Xi)\|^m_{L^p(\mathbb{R}^d)}. \end{aligned}$$

By Besov embedding, $\delta_0 \in \mathcal{B}^0_1 \hookrightarrow \mathcal{B}^{-d+d/q'}_{q'} = \mathcal{B}^{-dm/p}_{q'}$. Hence by Lemma 3.2$(i)$,

$$\mathbb{E}^s |\mathbb{E}^u [f(B_t, \Xi)]|^m \leq C \|\delta_0\|_{\mathcal{B}^{-dm/p}_{q'}}\, \sigma^{-dm/p}_{s,u,t}\, \|f(\cdot, \Xi)\|^m_{\mathcal{B}^\beta_p}\, \sigma^{\beta m}_{u,t}.$$

The fBm has the following local nondeterminism properties (see e.g. (C.3) and (C.5) in [1]): there exists $C_1, C_2 > 0$ such that

$$\sigma^2_{u,t} = C_1(t-u)^{2H} \quad \text{and} \quad \sigma^2_{s,u,t} \geq C_2(u-s)(t-s)^{-1+2H}. \tag{3.1}$$

It follows that

$$\mathbb{E}^s|\mathbb{E}^u[f(B_t,\Xi)]|^m \leq C\|\delta_0\|_{\mathcal{B}^{-dm/p}_{q'}} \|f(\cdot,\Xi)\|^m_{\mathcal{B}^\beta_p} (u-s)^{-\frac{dm}{2p}}(t-s)^{(1-2H)\frac{dm}{2p}} (t-u)^{H\beta m}.$$

We conclude by taking the expectation in the above inequality and raising both sides to the power $1/m$. □

### 3.2 Regularisation effect of the $d$-dimensional fBm

We use the stochastic sewing lemma of [24] (recalled in Lemma A.1) to establish the key regularisation result (Proposition 3.5) that will be used to prove existence of weak solutions. Note that the results in this subsection and the next one are similar to the one-dimensional framework developed in [1].

First we have the following lemma, which extends [1, Lemma D.2] to dimension $d \geq 1$. Its proof, which is also close to the proof of [1, Lemma D.2], is postponed to the Appendix A.1.

**Lemma 3.4.** *Let $\beta \in (-1/(2H), 0)$ such that $\beta - d/p \in (-1/H, 0)$. Let $m \in [2, \infty]$, $q \in [m, \infty]$ and assume that $p \in [q, +\infty]$. Then there exists a constant $C > 0$ such that for any $0 \leq S \leq T$, any $\mathcal{F}_S$-measurable random variable $\Xi$ in $\mathbb{R}^e$ and any bounded measurable function $f : \mathbb{R}^d \times \mathbb{R}^e \to \mathbb{R}^d$ fulfilling*

*(i)* $\mathbb{E}\left[\|f(\cdot,\Xi)\|^2_{\mathcal{C}^1}\right] < \infty$;

*(ii)* $\mathbb{E}\left[\|f(\cdot,\Xi)\|^q_{\mathcal{B}^\beta_p}\right] < \infty$,

*we have for any $(s,t) \in \Delta_{S,T}$ that*

$$\left\| \left(\mathbb{E}^S\Big|\int_s^t f(B_r,\Xi)\,dr\Big|^m\right)^{\frac{1}{m}}\right\|_{L^q} \leq C\,\|\|f(\cdot,\Xi)\|_{\mathcal{B}^\beta_p}\|_{L^q}\,(t-s)^{1+H(\beta-\frac{d}{p})}. \tag{3.2}$$

As a consequence of Lemma A.1 and Lemma 3.4, we get the following property of regularisation of the $d$-dimensional fBm. It can be compared to [1, Lemma 7.1], which is stated for one-dimensional processes in the sub-critical case only. The proof is postponed to Appendix A.2.

**Proposition 3.5.** *Let $m \in [2, \infty)$, $q \in [m, +\infty]$ and $p \in [q, +\infty]$.*

*(a) <u>The sub-critical case</u>: let $\beta \in (-1/(2H), 0)$ such that $\beta - d/p > -1/(2H)$. Let $\tau \in (0, 1)$ such that $H(\beta - d/p - 1) + \tau > 0$. There exists a constant $C > 0$ such that for any $f \in \mathcal{C}^\infty_b(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}^\beta_p$, any $\mathbb{R}^d$-valued stochastic process $(\psi_t)_{t\in[0,1]}$ adapted to $\mathbb{F}$, any $(S,T) \in \Delta_{0,1}$ and $(s,t) \in \Delta_{S,T}$ we have*

$$\begin{aligned}\left\| \left(\mathbb{E}^S\Big|\int_s^t f(B_r+\psi_r)\,dr\Big|^m\right)^{\frac{1}{m}}\right\|_{L^q} &\leq C\,\|f\|_{\mathcal{B}^\beta_p}(t-s)^{1+H(\beta-\frac{d}{p})} \\ &\quad + C\|f\|_{\mathcal{B}^\beta_p}[\psi]_{\mathcal{C}^\tau_{[S,T]}L^{m,q}}\,(t-s)^{1+H(\beta-\frac{d}{p}-1)+\tau}.\end{aligned} \tag{3.3}$$

*(b) <u>The critical case</u>: let $\beta - d/p = -1/(2H)$ and assume that $\beta > 1 - 1/(2H)$. There exists a constant $C > 0$ such that for any $f \in \mathcal{C}^\infty_b(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}^{\beta+1}_p$, any $\mathbb{R}^d$-valued stochastic process $(\psi_t)_{t\in[0,1]}$ adapted to $\mathbb{F}$, any $(S,T) \in \Delta_{0,1}$ and any $(s,t) \in \Delta_{S,T}$, we have*

$$\Big\|\int_s^t f(B_r+\psi_r)\,dr\Big\|_{L^m} \leq C\,\|f\|_{\mathcal{B}^\beta_p}\left(1 + \left|\log\frac{\|f\|_{\mathcal{B}^\beta_p}}{\|f\|_{\mathcal{B}^{\beta+1}_p}}\right|\right)\left(1 + [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^m}\right)(t-s)^{\frac{1}{2}}. \tag{3.4}$$

### 3.3 Tightness and stability

The proof of existence of a weak solution is based on a classical argument: first, we construct a tight approximating sequence of processes (Proposition 3.8), then we prove the stability, i.e. that any converging subsequence is a solution of the SDE (1.1) (Proposition 3.9).

First we need two *a priori* estimates, which are direct consequences of Proposition 3.5($a$): Lemma 3.6 (resp. Lemma 3.7), which extends [1, Lemma 7.3] (resp. [1, Lemma 7.4]) to dimension $d \geq 1$.

**Lemma 3.6.** *Assume that* (H1) *holds and let* $m \in [2, \infty)$. *There exists* $C > 0$*, such that, for any* $b \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\gamma$,

$$[X - B]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[0,1]} L^{m,\infty}} \leq C\,(1 + \|b\|^2_{\mathcal{B}_p^\gamma}), \tag{3.5}$$

*where* $X$ *is the strong solution to* (1.1) *with drift* $b$.

*Proof.* Without loss of generality, assume that $X_0 = 0$ and denote $K = X - B$. Then $[K]_{\mathcal{C}^\tau_{[0,1]} L^{m,\infty}}$ is finite for any $\tau \in (0, 1]$ as $|K_t - K_s| = |\int_s^t b(B_r + K_r)\, dr| \leq \|b\|_\infty |t - s|$.

We aim to apply Proposition 3.5($a$) with $m \in [2, \infty)$, $q = \infty$, $p = \infty$, $\beta = \gamma - \frac{d}{p}$ and $\tau = 1 + H(\gamma - d/p)$. The assumptions of Proposition 3.5($a$) are satisfied since $\tau - H > 1/2 - H/2 > 0$, thus $\tau \in (0, 1)$. In addition, by (H1), we have $H(\gamma - d/p) > H/2 - 1/2 > -1/2$ and $H(\gamma - d/p - 1) + \tau > 0$. Then we get

$$\begin{aligned}\|\mathbb{E}^s(|K_t - K_s|^m)^{\frac{1}{m}}\|_{L^\infty} &\leq C\,\|b\|_{\mathcal{B}_\infty^{\gamma-d/p}} \Big((t-s)^{1+H(\gamma-\frac{d}{p})} + [K]_{\mathcal{C}^\tau_{[s,t]} L^{m,\infty}} (t-s)^{1+H(\gamma-\frac{d}{p})+\tau-H}\Big)\\ &= C\|b\|_{\mathcal{B}_p^\gamma} (t-s)^{1+H(\gamma-\frac{d}{p})} \Big(1 + [K]_{\mathcal{C}^\tau_{[s,t]} L^{m,\infty}} (t-s)^{\tau-H}\Big). \end{aligned} \tag{3.6}$$

Choose $\ell = (4C\|b\|_{\mathcal{B}_p^\gamma})^{1/(H-\tau)}$ so that $C\|b\|_{\mathcal{B}_p^\gamma} \ell^{\tau-H} < 1/2$. Let $u \in [0, 1]$. Divide both sides in (3.6) by $(t - s)^{1+H(\gamma-d/p)}$ and take the supremum over $(s, t) \in \Delta_{u,(u+\ell)\wedge 1}$ to get

$$[K]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[u,(u+\ell)\wedge 1]} L^{m,\infty}} \leq \left(C\|b\|_{\mathcal{B}_p^\gamma} + \frac{1}{2}[K]_{\mathcal{C}^\tau_{[u,(u+\ell)\wedge 1]} L^{m,\infty}}\right),$$

and therefore

$$[K]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[u,(u+\ell)\wedge 1]} L^{m,\infty}} \leq 2C\|b\|_{\mathcal{B}_p^\gamma}.$$

The end of the proof consists in iterating the previous inequality in order to control the Hölder norm on the whole interval $[0, 1]$, and is completely identical to the proof of [1, Lemma 7.3]. ☐

**Lemma 3.7.** *Assume that* (H1) *holds and let* $b, h \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\gamma$. *Let* $X$ *be the strong solution to* (1.1) *with drift* $b$. *Let* $\delta \in (0, 1 + H(\gamma - d/p))$. *Then there exists a constant* $C > 0$ *which does not depend on* $X_0$, $b$ *and* $h$, *and a nonnegative random variable* $Z$ *which satisfies* $\mathbb{E}[Z] \leq C\|h\|_{\mathcal{B}_p^\gamma}(1 + \|b\|^2_{\mathcal{B}_p^\gamma})$ *such that for any* $(s, t) \in \Delta_{0,1}$,

$$\Big|\int_s^t h(X_r)\, dr\Big| \leq Z\, |t - s|^\delta. \tag{3.7}$$

The proof relies on Proposition 3.5($a$), Lemma 3.6 and Kolmogorov's continuity criterion. We omit it, as it is identical to the proof of [1, Lemma 7.4].

We now obtain tightness of the sequence that approximates $X$.

**Proposition 3.8.** *Assume that* (H1) *holds, let* $(b^n)_{n\in\mathbb{N}}$ *be a sequence of smooth bounded functions converging to* $b$ *in* $\mathcal{B}_p^{\gamma-}$. *For* $n \in \mathbb{N}$, *let* $X^n$ *be the strong solution to* (1.1) *with initial condition* $X_0$ *and drift* $b^n$. *Then there exists a subsequence* $(n_k)_{k\in\mathbb{N}}$ *such that* $(X^{n_k}, B)_{k\in\mathbb{N}}$ *converges weakly in the space* $[\mathcal{C}_{[0,1]}(\mathbb{R}^d)]^2$.

*Proof.* This short proof is close to the proof of [1, Proposition 7.5]. We reproduce it here for the reader's convenience.

Let $K^n_t := \int_0^t b^n(X^n_r)\,dr$. For $M > 0$ and some $\delta \in (0, 1 + H(\gamma - d/p))$, let

$$A_M := \{f \in \mathcal{C}_{[0,1]} : f(0) = 0,\ |f(t) - f(s)| \leq M(t-s)^\delta,\ \forall (s,t) \in \Delta_{0,1}\}.$$

By Arzelà-Ascoli's theorem, $A_M$ is compact in $\mathcal{C}_{[0,1]}$. Applying Lemma 3.7 to $h = b^n$, this gives a nonnegative random variable $Z^n$ such that $\mathbb{E}[Z^n] \leq C\|b^n\|_{\mathcal{B}^\gamma_p}(1 + \|b^n\|^2_{\mathcal{B}^\gamma_p})$ and (3.7) is satisfied. Thus by Markov's inequality we get

$$\begin{aligned}\mathbb{P}(K^n \notin A_M) &\leq \mathbb{P}(\exists (s,t) \in \Delta_{0,1} : |K^n_{s,t}| > M(t-s)^\delta) \\ &\leq \mathbb{P}(Z^n > M) \\ &\leq C \sup_{n\in\mathbb{N}} \|b^n\|_{\mathcal{B}^\gamma_p} \,(1 + \sup_{n\in\mathbb{N}} \|b^n\|^2_{\mathcal{B}^\gamma_p})\, M^{-1}.\end{aligned}$$

Hence, the sequence $(K^n)_{n\in\mathbb{N}}$ is tight in $\mathcal{C}_{[0,1]}$. So $(K^n, B)_{n\in\mathbb{N}}$ is tight in $(\mathcal{C}_{[0,1]})^2$. Thus by Prokhorov's Theorem, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $(K^{n_k}, B)_{k\in\mathbb{N}}$ converges weakly in the space $(\mathcal{C}_{[0,1]})^2$, and so does $(X^{n_k}, B)_{k\in\mathbb{N}}$. ☐

Finally, the stability is expressed in the following proposition, which extends [1, Proposition 7.7] to dimension $d \geq 1$.

**Proposition 3.9.** *Assume that* (H1) *holds and let* $(b^n)_{n\in\mathbb{N}}$ *be a sequence of smooth bounded functions converging to* $b$ *in* $\mathcal{B}^{\gamma-}_p$. *Let* $\hat{B}^n$ *have the same law as* $B$. *We consider* $\hat{X}^n$ *the strong solution to* (1.1) *for* $B = \hat{B}^n$, *initial condition* $X_0$ *and drift* $b^n$. *Assume that there exist stochastic processes* $\hat{X}, \hat{B} : [0,1] \to \mathbb{R}^d$ *such that* $(\hat{X}^n, \hat{B}^n)_{n\in\mathbb{N}}$ *converges to* $(\hat{X}, \hat{B})$ *on* $[\mathcal{C}_{[0,1]}(\mathbb{R}^d)]^2$ *in probability. Then* $\hat{X}$ *fulfills* (2.5) *and* (2.6) *from Definition 2.2 and for any* $m \in [2, \infty)$, *there exists* $C > 0$ *such that*

$$[\hat{X} - \hat{B}]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[0,1]} L^{m,\infty}} \leq C\,(1 + \sup_{n\in\mathbb{N}} \|b^n\|^2_{\mathcal{B}^\gamma_p}) < \infty. \tag{3.8}$$

The proof is postponed to Appendix A.3.

## 3.4 Proof of Theorem 2.3 *(a)*

Let $(b^n)_{n\in\mathbb{N}}$ be a sequence of smooth bounded functions converging to $b$ in $\mathcal{B}^{\gamma-}_p$. By Proposition 3.8, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $(X^{n_k}, B)_{k\in\mathbb{N}}$ converges weakly in $(\mathcal{C}_{[0,1]}(\mathbb{R}^d))^2$. Without loss of generality, we assume that $(X^n, B)_{n\in\mathbb{N}}$ converges weakly. By the Skorokhod representation Theorem, there exists a sequence of random variables $(\hat{X}^n, \hat{B}^n)_{n\in\mathbb{N}}$ defined on a common probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, such that

$$\mathrm{Law}(\hat{X}^n, \hat{B}^n) = \mathrm{Law}(X^n, B),\ \forall n \in \mathbb{N}, \tag{3.9}$$

and $(\hat{X}^n, \hat{B}^n)$ converges a.s. to some $(\hat{X}, \hat{B})$ in $(\mathcal{C}_{[0,1]}(\mathbb{R}^d))^2$. As $X^n$ solves (1.1) with drift $b^n$, we know by (3.9) that $\hat{X}^n$ also solves (1.1) with drift $b^n$ and $\hat{B}^n$ instead of $B$. As $X^n$ is a strong solution, we have that $X^n$ is adapted to $\mathbb{F}^B$. Hence by (3.9), we know that $\hat{X}^n$ is adapted to $\mathbb{F}^{\hat{B}^n}$ as the conditional laws of $\hat{X}^n$ and $X^n$ agree, and therefore it is a strong solution to (1.1) with $\hat{B}^n$ instead of $B$.

By Proposition 3.9, we know that $\hat{X}$ fulfills (2.5) and (2.6) from Definition 2.2 with $\hat{B}$ instead of $B$ and it is adapted with respect to the filtration $\hat{\mathbb{F}}$ defined by $\hat{\mathcal{F}}_t := \sigma(\hat{X}_s, \hat{B}_s, s \in [0,t])$. It remains to check that $\hat{B}$ is an $\hat{\mathbb{F}}$-fBm, which is completely analogous to the one-dimensional case treated in the proof of Theorem 2.8 in [1]. Hence $\hat{X}$ is a weak solution.

Finally, (3.8) gives that

$$[\hat{X} - \hat{B}]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[0,1]} L^{m,\infty}} < \infty,$$

which concludes the proof.

## 3.5 Proof of Theorem 2.3*(b)*

Although it will be proven in the next sections, we will use Corollary 2.6 to prove Theorem 2.3*(b)*.

Assuming (H2), we let $(X, B)$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a weak solution to (1.1) given by Theorem 2.3*(a)*. On this probability space and with the same fBm $B$, we define the tamed Euler scheme $(X^{h,n})_{h>0,n\in\mathbb{N}}$. As in Corollary 2.6, we let $b^n = G_{\frac{1}{n}} b$, $n_h = \lfloor h^{-\frac{1}{1-\gamma+d/p}} \rfloor$ and consider the scheme $(X^{h,n_h})_{h\in(0,1)}$.

First, observe that $X^{h,n_h}$ is $\mathbb{F}^B$-adapted. In view of (2.9), $X_t^{h,n_h}$ converges to $X_t$ in $L^m$, for each $t \in [0,1]$. Hence $X_t$ is $\mathcal{F}_t^B$-measurable and $X$ is therefore a strong solution.

As for the uniqueness, if $X$ and $Y$ are two strong solutions to (1.1) with the same fBm $B$, such that $[X-B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{2,\infty}} < \infty$ and $[Y-B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{2,\infty}} < \infty$, then by Corollary 2.6, $X^{h,n_h} = Y^{h,n_h}$ approximates both $X$ and $Y$. So $X$ and $Y$ are modifications of one another. Since they are continuous processes, they are indistinguishable. This proves uniqueness in the class of solutions $X$ such that $[X-B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{2,\infty}} < \infty$. In Appendix B, for any $\eta \in (0,1)$ and in the sub-critical regime $\gamma - d/p > 1 - 1/(2H)$, we extend the uniqueness result to the class of solutions $X$ such that $[X-B]_{\mathcal{C}^{H(1-\gamma+d/p)+\eta}_{[0,1]} L^{m,\infty}} < \infty$.

# 4 Convergence of the tamed Euler scheme

Let $\gamma$ and $p$ satisfying (H2) and $b \in \mathcal{B}_p^\gamma$. By a Besov embedding, we have $\mathcal{B}_p^\gamma \hookrightarrow \mathcal{B}_q^{\gamma-\frac{d}{p}-\frac{d}{q}}$ for any $q \geq p$. Setting $\tilde{\gamma} = \gamma - \frac{d}{p} + \frac{d}{q}$ and $\tilde{p} = q$, we have $b \in \mathcal{B}_{\tilde{p}}^{\tilde{\gamma}}$ and $\gamma - d/p = \tilde{\gamma} - d/\tilde{p}$, so that (H2) is still satisfied in $\mathcal{B}_{\tilde{p}}^{\tilde{\gamma}}$. Hence without any loss of generality, considering a smaller $\gamma$, we can always assume that $p$ is as large as we want. For the proof, we assume that $p \geq m$. This allows us in particular to apply the regularisation lemmas (see Proposition 3.5 and Section 5.1).

## 4.1 Proof of Theorem 2.5

First, we skip the first point (Theorem 2.5$(a)$) about the regularity of the scheme $X^{h,n}$. It is proven in Corollary 5.9, and follows from several technical lemmas presented in Section 5.

Now we prove Theorem 2.5$(b)$ and $(c)$. Let $(X, B)$ be a weak solution to (1.1) defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. On this probability space and with the same fBm $B$, we define the tamed Euler scheme $(X^{h,n})_{h>0,n\in\mathbb{N}}$.

For all $t > 0$, recall from (2.5) that $K_t := X_t - B_t - X_0$ and define

$$K_t^n := \int_0^t b^n(X_r)\, dr \quad \text{and} \quad K_t^{h,n} := \int_0^t b^n(X_{r_h}^{h,n})\, dr. \tag{4.1}$$

With these notations in mind, we set the notation for the error as

$$\mathcal{E}_t^{h,n} := X_t - X_t^{h,n}, \quad t \geq 0.$$

Let $0 \leq S \leq T \leq 1$. The error is decomposed as

$$\begin{aligned} [\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\varsigma}_{[S,T]} L^m} &\leq [K-K^n]_{\mathcal{C}^{\frac{1}{2}-\varsigma}_{[S,T]} L^m} + [K^n - K^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\varsigma}_{[S,T]} L^m} \\ &= [K-K^n]_{\mathcal{C}^{\frac{1}{2}-\varsigma}_{[S,T]} L^m} + [E^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\varsigma}_{[S,T]} L^m} \\ &\leq [K-K^n]_{\mathcal{C}^{\frac{1}{2}-\varsigma}_{[S,T]} L^m} + [E^{1,h,n}]_{\mathcal{C}^{\frac{1}{2}-\varsigma}_{[S,T]} L^m} + [E^{2,h,n}]_{\mathcal{C}^{\frac{1}{2}-\varsigma}_{[S,T]} L^m}, \end{aligned} \tag{4.2}$$

where $\zeta = 0$ in the subcritical case, and for all $s < t$ we denote

$$\begin{aligned}
E^{h,n}_{s,t} &:= K^n_t - K^n_s - (K^{h,n}_t - K^{h,n}_s),\\
E^{1,h,n}_{s,t} &:= \int_s^t b^n(X_0 + K_r + B_r) - b^n(X_0 + K^{h,n}_r + B_r)\, dr,\\
E^{2,h,n}_{s,t} &:= \int_s^t b^n(X_0 + K^{h,n}_r + B_r) - b^n(X_0 + K^{h,n}_{r_h} + B_{r_h})\, dr.
\end{aligned} \tag{4.3}$$

We also denote

$$\epsilon(h,n) := [K - K^n]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} + [E^{2,h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m}. \tag{4.4}$$

In order to prove Theorem 2.5($b$) and ($c$), we will provide bounds on the quantities that appear in the right-hand side of (4.2). The bound on $E^{1,h,n}$ is stated and proven in Section 5, see Corollary 5.3 for the sub-critical case, and in Proposition 5.4 for the critical case. The bound on $E^{2,h,n}$ is proven in Corollary 5.13 for both cases. We now prove the bounds on $K - K^n$ in both cases.

**Bound on $K - K^n$.** Let $k, n \in \mathbb{N}$.

First, in the case $\gamma - d/p > 1 - 1/(2H)$, we apply Proposition 3.5($a$) with $f = b^k - b^n$, $\tau = 1/2 + H$, $\beta = \gamma - 1$ and $\psi = X - B$. Using $[X - B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^m} < \infty$, it comes that for any $(s,t) \in \Delta_{S,T}$,

$$\| K^k_t - K^k_s - K^n_t + K^n_s \|_{L^m} \le C \| b^k - b^n \|_{\mathcal{B}^{\gamma-1}_p} (t-s)^{1 + H(\gamma - 1 - \frac{d}{p})}.$$

Hence $(K^k_t - K^k_s)_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^m(\Omega)$ and therefore it converges. We also know by definition of $X$ that $K^k_t - K^k_s$ converges in probability to $K_t - K_s$. Thus $K^k_t - K^k_s$ converges in $L^m$ to $K_t - K_s$. Now by the convergence of $b^k$ to $b$ in $\mathcal{B}^{\gamma-1}_p$, we get

$$\| K_t - K_s - K^n_t + K^n_s \|_{L^m} \le C \| b - b^n \|_{\mathcal{B}^{\gamma-1}_p} (t-s)^{1 + H(\gamma - 1 - \frac{d}{p})}.$$

Dividing by $(t-s)^{\frac{1}{2}}$ and taking the supremum over $(s,t)$ in $\Delta_{S,T}$ (recall that $\frac{1}{2} + H(\gamma - 1 - d/p) \ge 0$), we get that

$$[K - K^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]} L^m} \le C \| b - b^n \|_{\mathcal{B}^{\gamma-1}_p}. \tag{4.5}$$

Now in the critical case, i.e. with $\gamma - d/p = 1 - 1/(2H)$ and $\gamma > 1 - 1/(2H)$, we will apply Proposition 3.5($b$) with $f = b^k - b^n$, $\beta = \gamma - 1$ and $\psi = X - B$. Since $(b^n)$ converges to $b$ in $\mathcal{B}^{\gamma-}_p$, there is $\| b^k - b^n \|_{\mathcal{B}^{\gamma}_p} \le 2 \| b \|_{\mathcal{B}^{\gamma}_p} \vee 1$, and therefore

$$\begin{aligned}
\left| \log \frac{\| b^k - b^n \|_{\mathcal{B}^{\gamma-1}_p}}{\| b^k - b^n \|_{\mathcal{B}^{\gamma-}_p}} \right| &\le \log(2 \| b \|_{\mathcal{B}^{\gamma}_p} \vee 1) + |\log(\| b^k - b^n \|_{\mathcal{B}^{\gamma-1}_p})| \\
&\le C(1 + |\log(\| b^k - b^n \|_{\mathcal{B}^{\gamma-1}_p})|).
\end{aligned}$$

Besides, $[X - B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^m} < \infty$, hence for any $(s,t) \in \Delta_{S,T}$, Proposition 3.5($b$) reads

$$\| K^k_t - K^k_s - K^n_t + K^n_s \|_{L^m} \le C \| b^k - b^n \|_{\mathcal{B}^{\gamma-1}_p} (1 + |\log \| b^k - b^n \|_{\mathcal{B}^{\gamma-1}_p}|)(t-s)^{\frac{1}{2}}.$$

As in the sub-critical case, we deduce that

$$[K - K^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]} L^m} \le C \| b - b^n \|_{\mathcal{B}^{\gamma-1}_p} (1 + |\log \| b - b^n \|_{\mathcal{B}^{\gamma-1}_p}|). \tag{4.6}$$

**Bound on $E^{1,h,n}$.** Recall that $\mathcal{E}^{h,n} = K - K^{h,n}$ and that $X$ is a weak solution constructed in Theorem 2.3(a) that satisfies $[X-B]_{\mathcal{C}^{1/2+H}_{[S,T]}L^m} \leq [X-B]_{\mathcal{C}^{1/2+H}_{[0,1]}L^{m,\infty}} < \infty$.

In the sub-critical case, we have from Corollary 5.3 that there exists $C > 0$ such that for any $(s,t) \in \Delta_{S,T}$, any $n \in \mathbb{N}$ and $h \in (0,1)$,

$$\|E^{1,h,n}_{s,t}\|_{L^m} \leq C\Big([\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + \|\mathcal{E}^{h,n}_S\|_{L^m}\Big)(t-s)^{1+H(\gamma-1-\frac{d}{p})}.$$

Divide by $(t-s)^{1/2}$ and take the supremum over $(s,t) \in \Delta_{S,T}$ to get

$$[E^{1,h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\Big([\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + \|\mathcal{E}^{h,n}_S\|_{L^m}\Big)(T-S)^{\frac{1}{2}+H(\gamma-1-\frac{d}{p})}. \tag{4.7}$$

In the critical case, for $\mathcal{D}$ a sub-domain of $(0,1)\times\mathbb{N}$ such that (H3) holds, Proposition 5.4 yields the existence of $\ell_0 > 0$ such that if $T-S \leq \ell_0$, then for any $(s,t) \in \Delta_{S,T}$,

$$\begin{aligned}\|E^{1,h,n}_{s,t}\|_{L^m} \leq \mathbf{M}\left(1 + \Big|\log \frac{T^H(1+[K^{h,n}]_{\mathcal{C}^{1/2+H}_{[S,T]}L^m})}{\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)}\Big|\right)\Big(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)\Big)(t-s)\\ + \mathbf{M}\left(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m}\right)(t-s)^{\frac{1}{2}}.\end{aligned}$$

Since $\mathcal{D}$ satisfies (H3), we have from Corollary 5.9 that

$$\sup_{(h,n)\in\mathcal{D}} [K^{h,n}]_{\mathcal{C}^{\frac{1}{2}+H}_{[0,1]}L^m} < \infty.$$

It follows that

$$\begin{aligned}\|E^{1,h,n}_{s,t}\|_{L^m} \leq \mathbf{M}\Big(1 + |\log(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n))|\Big)\Big(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)\Big)(t-s)\\ + C\left((1+|\log T|(t-s)^{\frac{1}{2}})(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)) + [\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m}\right)(t-s)^{\frac{1}{2}}.\end{aligned}$$

Now use that $1 \geq T \geq t-s$ to deduce that $|\log T|(t-s)^{\frac{1}{2}}$ is bounded on the set $\{(s,t,T) : T \in (0,1] \text{ and } s < t \leq T\}$. Since $\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m} \leq \|\mathcal{E}^{h,n}_S\|_{L^m}+[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m}$, we get using $\mathbf{M}(t-s) \leq C(t-s)^{\frac{1}{2}}$,

$$\begin{aligned}\|E^{1,h,n}_{s,t}\|_{L^m} \leq \mathbf{M}\,|\log(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n))|\Big(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)\Big)(t-s)\\ + C\left(\|\mathcal{E}^{h,n}_S\|_{L^m}+[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} + \epsilon(h,n)\right)(t-s)^{\frac{1}{2}}.\end{aligned}$$

Divide by $(t-s)^{1/2-\zeta}$ and take the supremum over $(s,t) \in \Delta_{S,T}$ to get

$$\begin{aligned}[E^{1,h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} \leq \mathbf{M}\Big(|\log(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n))|\Big)\Big(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)\Big)(T-S)^{\frac{1}{2}+\zeta}\\ + C\left(\|\mathcal{E}^{h,n}_S\|_{L^m}+[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} + \epsilon(h,n)\right)(T-S)^{\zeta}.\end{aligned} \tag{4.8}$$

**Bound on $E^{2,h,n}$.** By Corollary 5.13, we have the following bound for $\varepsilon \in (0,\frac{1}{2})$, and $(s,t) \in \Delta_{S,T}$

$$\|E^{2,h,n}_{s,t}\|_{L^m} \leq C\left(\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\right)(t-s)^{\frac{1}{2}}.$$

Dividing by $(t-s)^{\frac{1}{2}}$ and taking the supremum over $(s,t)$ in $\Delta_{S,T}$, we get

$$[E^{2,h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\left(\|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}\right). \tag{4.9}$$

This is where we avoid using Girsanov's theorem and rely instead on a bound that involves the $\mathcal{C}^1$ norm of $b^n$. This simply comes from estimates when $t-s \leq h$ of the form $|\int_s^t f(\psi_r + B_r) - f(\psi_r + B_{r_h})\,dr| \lesssim \|f\|_{\mathcal{C}^1}\,(t-s)\,h^{H-}$, at a scale where the discretised noise cannot regularise anymore. More rigorously, the previous bound is again obtained by a stochastic sewing argument.

**Conclusion in the sub-critical case.** Using (4.7) in (4.2), and recalling the definition of $\epsilon(h,n)$ in (4.4), we get

$$[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq \epsilon(h,n) + C\Big([\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + \|\mathcal{E}^{h,n}_S\|_{L^m}\Big)(T-S)^{\frac{1}{2}+H(\gamma-1-\frac{d}{p})}.$$

Hence for $T-S \leq (2C)^{-1/(1/2+H(\gamma-1-d/p))} =: \ell_0$, we get

$$[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq 2\epsilon(h,n) + \|\mathcal{E}^{h,n}_S\|_{L^m}. \tag{4.10}$$

Then the inequality

$$\|\mathcal{E}^{h,n}_S\|_{L^m} \leq \|\mathcal{E}^{h,n}_{S-\ell_0}\|_{L^m} + \ell_0^{\frac{1}{2}}[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S-\ell_0,S]}L^m}$$

can be plugged in (4.10) and iterated until $S-k\ell_0$ is smaller than 0 for $k\in\mathbb{N}$ large enough. It follows that

$$[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]}L^m} \leq C\epsilon(h,n),$$

and in view of (4.4), (4.5) and (4.9), we obtain the result (2.7) of Theorem 2.5($b$).

**Conclusion in the critical case.** Using (4.8) in (4.2), we get that if $T-S\leq \ell_0$,

$$\begin{aligned}
[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} &\leq [K-K^n]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} + [E^{2,h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} \\
&\quad + \mathbf{M}\Big(|\log(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n))|\Big)\Big(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)\Big)(T-S)^{\frac{1}{2}+\zeta} \\
&\quad + C\Big(\|\mathcal{E}^{h,n}_S\|_{L^m}+[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} + \epsilon(h,n)\Big)(T-S)^{\zeta}.
\end{aligned}$$

We observe that $[K-K^n]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} + [E^{2,h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} \leq (T-S)^{\zeta}\,\epsilon(h,n)$. Let $\ell>0$ satisfying

$$\ell < (C)^{-\frac{1}{\zeta}} \wedge 1 \wedge \ell_0. \tag{4.11}$$

Passing the term $[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m}$ from the r.h.s to the l.h.s, we get for any $S<T$ such that $T-S\leq\ell$

$$\begin{aligned}
[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m} &\leq \frac{1+C}{1-C\ell^{\zeta}}(\epsilon(h,n) + \|\mathcal{E}^{h,n}_S\|_{L^m})\,(T-S)^{\zeta} \\
&\quad + \frac{\mathbf{M}}{1-C\ell^{\zeta}}\,|\log(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n))|\Big(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)\Big)(T-S)^{\frac{1}{2}+\zeta}.
\end{aligned} \tag{4.12}$$

Hence denoting $C_1 = \frac{1+C}{1-C\ell^{\zeta}}$ and $\mathbf{C}_2 = \frac{\mathbf{M}}{1-C\ell^{\zeta}}$, we have for $T-S\leq\ell$,

$$\begin{aligned}
\|\mathcal{E}^{h,n}_T - \mathcal{E}^{h,n}_S\|_{L^m} &\leq C_1\Big(\epsilon(h,n) + \|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}\Big)(T-S)^{\frac{1}{2}} \\
&\quad + \mathbf{C}_2\Big(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)\Big)|\log(\|\mathcal{E}^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n))|\,(T-S).
\end{aligned} \tag{4.13}$$

We will now rely on the following technical lemma to conclude, which is a quantitative analogue of [3, Prop. 3.6] and [1, Prop. 6.1] (see in particular the use of Equation (6.9) in the latter result). We postpone the proof of this lemma to Subsection 4.2.

**Lemma 4.1.** *Let $(E, \|\cdot\|)$ be a normed vector space. For $\ell, C_1, C_2 > 0$ and $\eta \in (0,1)$ we consider the set $\mathcal{R}(\eta, \ell, C_1, C_2)$ of functions defined from $[0,1]$ to $E$ characterised as follows: $f \in \mathcal{R}(\eta, \ell, C_1, C_2)$ if $f$ is bounded, $f_0 = 0$ and for any $s \le t \in [0,1]$ such that $t - s \le \ell$,*

$$\|f_t - f_s\| \le C_1 \left(\|f\|_{L^\infty_{[s,t]}E} + \eta\right) (t-s)^{\frac{1}{2}} + C_2 \left(\|f\|_{L^\infty_{[s,t]}E} + \eta\right) \left|\log\left(\|f\|_{L^\infty_{[s,t]}E} + \eta\right)\right| (t-s). \tag{4.14}$$

*Then for any $\delta \in (0, e^{-C_2})$, there exists $\bar\eta \equiv \bar\eta(C_1, C_2, \ell, \delta)$ such that for any $\eta < \bar\eta$ and any $f \in \mathcal{R}(\eta, \ell, C_1, C_2)$,*

$$\|f\|_{L^\infty_{[0,1]}E} \le \eta^{e^{-C_2} - \delta}.$$

Let $\delta \in (0, \frac{e^{-\mathbf{M}}}{4})$. Recalling that $\mathbf{C}_2 = \frac{\mathbf{M}}{1 - C\ell^\zeta}$, let us also choose $\ell$ which still satisfies (4.11) and which is small enough in order to have $e^{-\mathbf{C}_2} \ge e^{-\mathbf{M}} - \delta \ge \delta$.

We now apply Lemma 4.1 with $E = L^m$, $\ell = \ell$, $C_1 = C_1$, $C_2 = \mathbf{C}_2$. By (4.13), we have that $\mathcal{E}^{h,n}$ belongs to $\mathcal{R}(\epsilon(h,n), \ell, C_1, \mathbf{C}_2)$ for $(h,n) \in \mathcal{D}$. Therefore, there exists $\bar\epsilon \equiv \bar\epsilon(C_1, \mathbf{C}_2, \ell, \delta)$ such that if $\epsilon(h,n) < \bar\epsilon$, we have

$$\|\mathcal{E}^{h,n}\|_{L^\infty_{[0,1]}L^m} \le \epsilon(h,n)^{e^{-\mathbf{C}_2} - \delta} \le \epsilon(h,n)^{e^{-\mathbf{M}} - 2\delta}.$$

Let $\epsilon \equiv \epsilon(C_1, \mathbf{M}, \ell, \delta, \zeta) < \bar\epsilon \wedge 1$ such that $\epsilon^{e^{-\mathbf{M}} - 2\delta} < \frac{e^{-1}}{2}$. Then, if $\epsilon(h,n) < \epsilon$, we have

$$\|\mathcal{E}^{h,n}\|_{L^\infty_{[0,1]}L^m} + \epsilon(h,n) \le \epsilon(h,n)^{e^{-\mathbf{M}} - 2\delta} + \epsilon(h,n) < 2\epsilon(h,n)^{e^{-\mathbf{M}} - 2\delta} < e^{-1}.$$

Since $x \mapsto x|\log(x)|$ is increasing over $(0, e^{-1})$, in view of (4.12), we have that over any interval $I$ of size $\ell$,

$$[\mathcal{E}^{h,n}]_{\mathcal{C}_I^{\frac{1}{2}-\zeta} L^m} \le C\, \epsilon(h,n)^{(e^{-\mathbf{M}} - 2\delta)} \left(1 + |\log(\epsilon(h,n))|\right).$$

Since $\ell$ is fixed independently of $\epsilon(h,n)$, summing at most $\frac{1}{\ell}$ of these bounds, we get that if $\epsilon(h,n) < \epsilon$

$$[\mathcal{E}^{h,n}]_{\mathcal{C}_{[0,1]}^{\frac{1}{2}-\zeta} L^m} \le C\, \epsilon(h,n)^{(e^{-\mathbf{M}} - 2\delta)} \left(1 + |\log(\epsilon(h,n))|\right) \le C\, \epsilon(h,n)^{(e^{-\mathbf{M}} - 4\delta)}.$$

From Corollary 5.9 and the property $[K]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{m,\infty}} < \infty$, we have that under (H3), $\sup_{h,n \in \mathcal{D}} [\mathcal{E}^{h,n}]_{\mathcal{C}_{[0,1]}^{\frac{1}{2}-\zeta} L^m} \le [K]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{m,\infty}} + \sup_{h,n \in \mathcal{D}} [K^{h,n}]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{m,\infty}} < \infty$. It follows that there exists a constant $C$ such that for all $(h,n) \in \mathcal{D}$,

$$[\mathcal{E}^{h,n}]_{\mathcal{C}_{[0,1]}^{\frac{1}{2}-\zeta} L^m} \le C\, \epsilon(h,n)^{(e^{-\mathbf{M}} - 4\delta)}.$$

In view of (4.4), we obtain (2.8) from Theorem 2.5$(c)$.

**Remark 4.2.** *The real-valued function $g_t = \eta^{e^{-Ct}} - \eta$ solves the differential equation*

$$g_t' = -C\,(g_t + \eta)\, \log(g_t + \eta), \ \ \forall t \in [0,1].$$

*Thus the result of Lemma 4.1 seems close to give an optimal bound. The term with the factor $C_1$ is only a small perturbation, that is why we do not keep track of the second constant in the bound (4.8) of $E^{1,h,n}$.*

### 4.2 Proof of Lemma 4.1

Let $\delta \in (0, \frac{1}{2}e^{-C_2})$. There exists $a > 1$ such that $e^{-C_2 \frac{a \log a}{a-1}} = e^{-C_2} - \delta$. Let $\varepsilon \equiv \varepsilon(C_2, \delta) \in (0,1)$ small enough such that $e^{-C_2 \frac{a \log a}{(a-1)(1-\varepsilon)}} \geq e^{-C_2} - 2\delta$. Denote also $\alpha := 1 - e^{-C_2 \frac{a \log a}{(a-1)(1-\varepsilon)}}$. Now for $\eta \in (0,1)$ and $f \in \mathcal{R}(\eta, \ell, C_1, C_2)$, define the following increasing sequence: $t_0 = 0$ and for $k \in \mathbb{N}$,

$$t_{k+1} = \begin{cases} \inf\{t > t_k : \ \eta + \|f_t\| \geq a^{k+1}\,\eta\} \wedge 1 & \text{if } t_k < 1, \\ 1 & \text{if } t_k = 1, \end{cases}$$

with the convention that $\inf \emptyset = +\infty$. In view of (4.14) and of the boundedness of $f$, the mapping $t \mapsto \|f_t\|$ is continuous. In particular, and by definition of the sequence $(t_k)$, we deduce that for any $k$,

$$\|f\|_{L^\infty_{[0,t_k]}E} \leq a^k \eta - \eta \leq a^k \eta.$$

Let

$$N = \left\lfloor -\alpha \frac{\log(\eta)}{\log(a)} \right\rfloor - 1, \tag{4.15}$$

and let $\bar{\eta}_0 \equiv \bar{\eta}_0(C_2, \delta)$ be such that for $\eta < \bar{\eta}_0$, we have $N \geq 1$. We shall prove the following statement:

$$\text{There exists } \bar{\eta} \equiv \bar{\eta}(C_1, C_2, \ell, \delta) \text{ such that for any } \eta < \bar{\eta} \text{ and } f \in \mathcal{R}(\eta, \ell, C_1, C_2),\ t_{N+1} = 1. \tag{4.16}$$

Observe that if (4.16) holds true, then for $\eta < \bar{\eta}$ and $f \in \mathcal{R}(\eta, \ell, C_1, C_2)$, we have

$$\|f\|_{L^\infty_{[0,1]}E} \leq a^{N+1}\eta \leq \eta^{1-\alpha} \leq \eta^{(e^{-C_2} - 2\delta)}$$

and the lemma is proven.

Let us now prove the statement (4.16). Fix $\eta < \bar{\eta}_0$ and $f \in \mathcal{R}(\eta, \ell, C_1, C_2)$. Let

$$N_0 = \inf\{k \in \mathbb{N} : \ t_{k+1} = 1\}.$$

We aim to prove that $N_0 \leq N$, so that we will have indeed $t_{N+1} = 1$. First, if $N_0 = 0$, we have obviously that $N \geq N_0$. Assume now that $N_0 \geq 1$. For any $k \leq N_0 - 1$, we have $\eta + \|f_{t_k}\| = a^k\,\eta$ and $\eta + \|f_{t_{k+1}}\| = a^{k+1}\,\eta$, which implies that $\|f_{t_{k+1}} - f_{t_k}\| \geq (a^{k+1} - a^k)\eta$. Consider two cases:

(1) If $t_{k+1} - t_k \leq \ell$, then one can apply (4.14), using $\|f_{t_{k+1}}\| = \|f\|_{L^\infty_{[t_k, t_{k+1}]}E}$, to get

$$(a^{k+1} - a^k)\eta \leq C_1\, a^{k+1}\eta\,(t_{k+1} - t_k)^{\frac{1}{2}} + C_2\, a^{k+1}\,\eta\,|\log(a^{k+1}\,\eta)|\,(t_{k+1} - t_k).$$

(2) If $t_{k+1} - t_k > \ell$, then we split the interval $[t_k, t_{k+1}]$ into at most $\lfloor \frac{1}{\ell} \rfloor$ intervals of size $\ell$, that we denote $[\beta_k^j, \beta_k^{j+1}]$. We can apply (4.14) over each such interval to get that

$$\begin{aligned}(a^{k+1} - a^k)\eta \leq C_1 \sum_{j=0}^{\lfloor \frac{1}{\ell} \rfloor} (\|f\|_{L^\infty_{[\beta_k^j, \beta_k^{j+1}]}E} + \eta)\,(\beta_k^{j+1} - \beta_k^j)^{\frac{1}{2}} \\ + C_2 \sum_{j=0}^{\lfloor \frac{1}{\ell} \rfloor} (\|f\|_{L^\infty_{[\beta_k^j, \beta_k^{j+1}]}E} + \eta)\,|\log(\|f\|_{L^\infty_{[\beta_k^j, \beta_k^{j+1}]}E} + \eta)|\,(\beta_k^{j+1} - \beta_k^j).\end{aligned}$$

By definition of the sequence $(t_k)_{k \in \mathbb{N}}$, we know that $\eta + \|f\|_{L^\infty_{[\beta_k^j, \beta_k^{j+1}]}E} \leq a^{k+1}\eta$. Moreover for $k \leq N$, there is $a^{k+1}\eta \leq a^{N+1}\eta \leq \eta^{e^{-C_2} - \delta}$. Therefore, for $\bar{\eta}_1 \equiv \bar{\eta}_1(C_2, \delta) \leq \bar{\eta}_0$ small enough, we have that $a^{k+1}\eta \le e^{-1}$ for any $\eta < \bar{\eta}_1$ and $k \le N$. Since the mapping $x \mapsto x\,|\log x|$ is nondecreasing on the interval $[0, e^{-1}]$, we get that

$$(\|f\|_{L^\infty_{[\beta_k^j,\beta_k^{j+1}]}E}+\eta)\,|\log(\|f\|_{L^\infty_{[\beta_k^j,\beta_k^{j+1}]}E}+\eta)|\le a^{k+1}\eta\,|\log(a^{k+1}\eta)|.$$

Then, by Cauchy-Schwarz on the first term, we write

$$(a^{k+1}-a^k)\eta \le C_1\, a^{k+1}\eta\sqrt{\frac{1}{\ell}}(t_{k+1}-t_k)^{\frac{1}{2}} + C_2\, a^{k+1}\,\eta\,|\log(a^{k+1}\eta)|\,(t_{k+1}-t_k).$$

Hence in both cases ($t_{k+1}-t_k \le \ell$ or $t_{k+1}-t_k > \ell$), for any $\eta < \bar{\eta}_1$, there is

$$1 \le \frac{C_1}{\sqrt{\ell}}\frac{a}{a-1}\,(t_{k+1}-t_k)^{\frac{1}{2}} + C_2\frac{a}{a-1}\,|\log(a^{k+1}\eta)|\,(t_{k+1}-t_k).$$

Notice that the polynomial $C_2\frac{a}{a-1}\,|\log(a^{k+1}\eta)|\,X^2 + \frac{C_1}{\sqrt{\ell}}\frac{a}{a-1}\,X - 1$ has only one non-negative root. Thus

$$(t_{k+1}-t_k)^{\frac{1}{2}} \ge \frac{-\frac{a}{a-1}\frac{C_1}{\sqrt{\ell}} + \sqrt{(\frac{a}{a-1})^2\frac{C_1^2}{\ell} + 4\frac{C_2 a}{a-1}|\log(a^{k+1}\eta)|}}{2\frac{C_2 a}{a-1}|\log(a^{k+1}\eta)|}.$$

Then we have

$$(t_{k+1}-t_k) \ge \frac{2\left(\frac{C_1 a}{\sqrt{\ell}(a-1)}\right)^2 + 4\frac{C_2 a}{a-1}|\log(a^{k+1}\eta)| - \frac{2C_1 a}{\sqrt{\ell}(a-1)}\sqrt{\left(\frac{C_1 a}{\sqrt{\ell}(a-1)}\right)^2 + 4\frac{C_2 a}{a-1}|\log(a^{k+1}\eta)|}}{\left(\frac{2C_2 a}{a-1}\right)^2|\log(a^{k+1}\eta)|^2}.$$

Using the notation $C_a = \frac{C_2 a}{a-1}$ and the inequality $\sqrt{a+b} \le \sqrt{a}+\sqrt{b}$, we get that for any $\eta < \bar{\eta}_1$,

$$\begin{aligned}(t_{k+1}-t_k) &\ge \frac{C_1^2}{2C_2^2\ell|\log(a^{k+1}\eta)|^2} + \frac{1}{C_a|\log(a^{k+1}\eta)|} - \frac{C_1^2}{2C_2^2\ell|\log(a^{k+1}\eta)|^2} - \frac{C_1}{(\frac{a}{a-1})^{\frac{1}{2}}C_2^{\frac{3}{2}}\sqrt{\ell}\,|\log(a^{k+1}\eta)|^{\frac{3}{2}}}\\ &\ge \frac{1}{C_a|\log(a^{k+1}\eta)|} - \frac{C_1}{C_2^{\frac{3}{2}}\sqrt{\ell}\,|\log(a^{k+1}\eta)|^{\frac{3}{2}}}. \end{aligned}\tag{4.17}$$

Now we will show that for $N$ defined in (4.15), the sum from 0 to $N$ of the right-hand side of (4.17) is larger than 1, which implies that $N_0 \le N$ since $\sum_{k=0}^{N_0}(t_{k+1}-t_k) = 1$. Let us start with the second term in the above inequality. Notice that for $k \le N$, we always have $|\log(a^{k+1}\eta)|= |\log(\eta)|-(k+1)\log(a)$. Thus we get

$$\begin{aligned}\sum_{k=0}^{N-1}\frac{1}{|\log(a^{k+1}\eta)|^{\frac{3}{2}}} &\le \int_0^N \frac{1}{|\log(a^{x+1}\eta)|^{\frac{3}{2}}}\,dx\\ &= \frac{2}{\log(a)}\left(-\frac{1}{\sqrt{|\log(a\eta)|}} + \frac{1}{\sqrt{|\log(a^{N+1}\eta)|}}\right).\end{aligned}$$

We have $|\log(a^{N+1}\eta)|= |\log(\eta)|-\lfloor\alpha\frac{|\log(\eta)|}{\log(a)}\rfloor\log(a) \ge (1-\alpha)|\log(\eta)|= e^{-C_2\frac{a\log a}{(a-1)(1-\varepsilon)}}|\log(\eta)|$. So we have $|\log(a^{N+1}\eta)|\to\infty$ as $\eta\to 0$ and therefore,

$$\begin{aligned}\lim_{\eta\to 0}\sum_{k=0}^{N}\frac{1}{|\log(a^{k+1}\eta)|^{\frac{3}{2}}} &\le \lim_{\eta\to 0}\frac{2}{\log(a)}\Big(-\frac{1}{\sqrt{|\log(a\eta)|}} + \frac{1}{\sqrt{|\log(a^{N+1}\eta)|}}\Big) + \frac{1}{|\log(a^{N+1}\eta)|^{\frac{3}{2}}}\\ &= 0.\end{aligned}$$

On the other hand,

$$\sum_{k=0}^{N}\frac{1}{C_a|\log(a^{k+1}\eta)|}\geq\frac{1}{C_a}\int_{-1}^{N-1}\frac{1}{|\log(a^{x+1}\eta)|}\,dx=\frac{1}{\log(a)C_a}\log\left(\frac{|\log(\eta)|}{|\log(a^N\eta)|}\right)$$
$$=\frac{1}{\log(a)C_a}\log\left(\frac{|\log(\eta)|}{(1-\alpha)|\log(\eta)|+\alpha|\log(\eta)|-N\log(a)}\right).$$

We have $N+1=\lfloor\frac{\alpha|\log(\eta)|}{\log(a)}\rfloor\geq\frac{\alpha|\log(\eta)|}{\log(a)}-1$, thus $N\log(a)+2\log(a)\geq\alpha|\log(\eta)|$. Hence,

$$\geq\frac{1}{\log(a)C_a}\log\left(\frac{|\log(\eta)|}{(1-\alpha)|\log(\eta)|+2\log(a)}\right).$$

The right hand side converges to $\log(1/(1-\alpha))/(C_a\log(a))$ as $\eta$ goes to 0. Hence, going back to (4.17) and summing over $k\in[\![0,N-1]\!]$, we know that there exists $\bar\eta\equiv\bar\eta(C_1,C_2,\ell,\delta)\leq\bar\eta_1$ such that for $\eta<\bar\eta$ , we have

$$\sum_{k=0}^{N}(t_{k+1}-t_k)\geq\frac{1}{C_a\log(a)}\log\left(\frac{1}{1-\alpha}\right)(1-\varepsilon)=1=\sum_{k=0}^{N_0}(t_{k+1}-t_k).$$

It follows that $N_0\leq N$ and thus $t_{N+1}=1$. Hence (4.16) is true and we conclude that for $\eta<\bar\eta$, we have $\|f\|_{L^\infty_{[0,1]}E}\leq\eta^{e^{-C_2-2\delta}}$ .

## 4.3 Proof of Corollary 2.6

We will do the computations with $n_h=\lfloor h^{-\alpha}\rfloor$ for some $\alpha>0$ and prove that the upper bound on $[\mathcal{E}^{h,n_h}]_{\mathcal{C}^{1/2}_{[0,1]}L^m}$ given by Theorem 2.5 is minimized for $\alpha=1/(1-\gamma+d/p)$.

First, the inequalities (2.11) and (2.12) imply that $b^{n_h}$ satisfy (H3) with $\mathcal{D}=\{(h,n_h),h\in(0,1/2)\}$, any $\eta\in(0,H)$ and $\alpha\leq(2(H-\eta)+1)/(1-\gamma+d/p)$. Therefore, we deduce (2.13) from Theorem 2.5($a$).

**The sub-critical case.** In view of Lemma 3.2, $b^{n_h}$ satisfies (2.10), (2.11), (2.12). Then the result of Theorem 2.5($b$) reads

$$[\mathcal{E}^{h,n_h}]_{\mathcal{C}^{\frac12}_{[0,1]}L^m}\leq C\Big(\lfloor h^{-\alpha}\rfloor^{-\frac12}+\lfloor h^{-\alpha}\rfloor^{-\frac12(\gamma-\frac{d}{p})}h^{\frac12-\varepsilon}+\lfloor h^{-\alpha}\rfloor^{\frac12-(\gamma-\frac{d}{p})}h^{1-\varepsilon}\Big).$$

Since $-\frac12(\gamma-\frac{d}{p})>0$ and $\frac12-(\gamma-\frac{d}{p})>0$, we have

$$\lfloor h^{-\alpha}\rfloor^{\frac12-(\gamma-\frac{d}{p})}\leq h^{-\frac{\alpha}{2}}h^{\alpha(\gamma-\frac{d}{p})}\text{ and }\lfloor h^{-\alpha}\rfloor^{-\frac12(\gamma-\frac{d}{p})}\leq h^{\frac{\alpha}{2}(\gamma-\frac{d}{p})}.$$

Moreover, since $h\in(0,\frac12)$ and $\lfloor h^{-\alpha}\rfloor>h^{-\alpha}-1$, we have

$$\lfloor h^{-\alpha}\rfloor^{-\frac12}\leq(1-h^{\alpha})^{-\frac12}h^{\frac{\alpha}{2}}\leq\left(1-\frac{1}{2^\alpha}\right)^{-\frac12}h^{\frac{\alpha}{2}}\leq Ch^{\frac{\alpha}{2}}.$$

It follows that

$$[\mathcal{E}^{h,n}]_{\mathcal{C}^{\frac12}_{[0,1]}L^m}\leq C\Big(h^{\frac{\alpha}{2}}+h^{\frac{\alpha}{2}(\gamma-\frac{d}{p})}h^{\frac12-\varepsilon}+h^{-\frac{\alpha}{2}}h^{\alpha(\gamma-\frac{d}{p})}h^{1-\varepsilon}\Big).$$

Now we optimize over $\alpha$. Introduce the following functions:

$$f_1(\alpha)=\frac{\alpha}{2},\quad f_2(\alpha)=\frac{\alpha}{2}\left(\gamma-\frac{d}{p}\right)+\frac12\ \text{ and }\ f_3(\alpha)=\left(\gamma-\frac{d}{p}-\frac12\right)\alpha+1,\quad\alpha>0.$$

Observe that $f_1$ is increasing and $f_2,f_3$ are decreasing. Moreover, we have

$$f_1(\alpha)=f_2(\alpha)=f_3(\alpha)\Leftrightarrow\alpha^\star=\frac{1}{1-\gamma+d/p}.\tag{4.18}$$

It follows that the error is minimized at $\alpha=\alpha^\star$. Let $n_h=\lfloor h^{-\alpha^\star}\rfloor$. This yields rate of convergence of order $1/(2(1-\gamma+d/p))-\varepsilon$, which proves (2.14).

**The critical case.** Using (2.10), (2.11) and (2.12), the result of Theorem 2.5($c$) reads

$$[\mathcal{E}^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[0,1]}L^m} \leq C\Big(\lfloor h^{-\alpha}\rfloor^{-\frac{1}{2}}(1+|\log(\lfloor h^{-\alpha}\rfloor^{-\frac{1}{2}})|)+\lfloor h^{-\alpha}\rfloor^{-\frac{1}{2}(\gamma-\frac{d}{p})}h^{\frac{1}{2}-\varepsilon}+\lfloor h^{-\alpha}\rfloor^{\frac{1}{2}-(\gamma-\frac{d}{p})}h^{1-\varepsilon}\Big)^{e^{-M}-\delta}.$$

Optimising over $\alpha$ again, we find $\alpha^\star = 2H$. This yields

$$[\mathcal{E}^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[0,1]}L^m} \leq C\Big(h^{H-\varepsilon}|\log(h)|\Big)^{e^{-\mathbf{M}}-\delta},$$

which proves (2.15)

### 4.4 Proof of Corollary 2.9

Let $\varepsilon \in (0,1/2)$, then fix $\eta \in (0,1/(2H)-1)$ and $\delta \in (0,\frac{1}{2})$ such that $\eta+\delta=\varepsilon$. Then $b$ also belongs to $\mathcal{B}^{-\eta}_\infty$ and Theorem 2.3($b$) states that there exists a strong solution $X$ to (1.1) which satisfies $X-B \in \mathcal{C}^{1/2+H}_{[0,T]}L^{m,\infty}$, which is pathwise unique in the class of solutions that satisfy $X-B\in\mathcal{C}^{H+\eta}_{[0,T]}L^{2,\infty}$.

To prove the second part of the corollary, apply Theorem 2.5($b$) with $\gamma=-\eta$, $p=\infty$ and $\varepsilon=\delta$, to get that for $\mathcal{D}$ satisfying (H3), we have $\sup_{(h,n)\in\mathcal{D}}[X^{h,n}-B]_{\mathcal{C}^{1/2+H}_{[0,1]}L^m}<\infty$. Moreover, noting that $\|b^n-b\|_{\mathcal{B}^{-\eta-1}_\infty}\leq C\|b^n-b\|_{\mathcal{B}^{-1}_\infty}$, it comes that

$$[X-X^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]}L^m} \leq C\left(\|b^n-b\|_{\mathcal{B}^{-1}_\infty}+\|b^n\|_\infty h^{\frac{1}{2}-\delta}+\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\delta}\right).$$

Now take $n_h=\lfloor h^{-\alpha}\rfloor$ and $b^{n_h}=G_{1/n_h}b$ for some $\alpha>0$. Using (2.10), (2.11) and (2.12) as in Subsection 4.3 leads to $\sup_{h\in(0,1/2)}[X^{h,n_h}-B]_{\mathcal{C}^{1/2+H}_{[0,1]}L^m}<\infty$ and

$$[X-X^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]}L^m} \leq C\left(h^{\frac{\alpha}{2}}+h^{\frac{1}{2}-\delta-\frac{\alpha\eta}{2}}+h^{1-\delta-\frac{\alpha}{2}-\alpha\eta}\right).$$

Optimising over $\alpha$ as before, we find $\alpha^\star=1/(1+\eta)$, which yields a rate of convergence of order $\frac{1}{2(1+\eta)}-\delta$. Since $\frac{1}{2(1+\eta)}\geq 1/2-\eta$, it finally comes that

$$[X-X^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]}L^m} \leq C\,h^{\frac{1}{2}-\eta-\delta}=C\,h^{\frac{1}{2}-\varepsilon}.$$

## 5 Regularisation effect of fBm and discrete-time fBm

In this section, $X$ always denotes a weak solution to (1.1) with drift $b\in\mathcal{B}^\gamma_p$, with $\gamma\in\mathbb{R}$ and $p\in[1,\infty]$ satisfying (H2). For such $X$, recall that the process $K$ is defined by (2.5). Let $(b^n)_\mathbb{N}$ be a sequence of smooth functions that converges to $b$ in $\mathcal{B}^{\gamma-}_p$. For $n\in\mathbb{N}$ and $h\in(0,1)$, recall that $X^{h,n}$ denotes the tamed Euler scheme (1.2) and that the process $K^{h,n}$ is defined by (4.1).

### 5.1 Regularisation in the strong well-posedness regime

In this subsection, we state and prove the bound on $E^{1,h,n}$, first in the sub-critical regime (Proposition 5.2 and Corollary 5.3), then in the critical regime (Proposition 5.4).

Recall that by a Besov embedding, $\mathcal{B}^\gamma_p\hookrightarrow\mathcal{B}^{\gamma-d/p+d/\tilde{p}}_{\tilde{p}}$ for any $\tilde{p}\geq p$. Setting $\tilde{\gamma}=\gamma-d/p+d/\tilde{p}$, we have $b\in\mathcal{B}^{\tilde{\gamma}}_{\tilde{p}}$ and $\gamma-d/p=\tilde{\gamma}-d/\tilde{p}$. So, without any loss of generality, we can always assume that $p$ is as large as we want. In this subsection, $p\geq m\geq 2$.

Before giving the main estimates of this section, we will need the following corollary of Lemma 3.4.

**Corollary 5.1.** *Let $\beta \in (-1/(2H), 0)$ such that $\beta - d/p \in (-1/H, 0)$. Let $m \in [2, \infty)$ and assume that $p \in [m, +\infty]$. Let $\lambda, \lambda_1, \lambda_2 \in (0, 1]$ and assume that $\beta > -1/(2H) + \lambda$ and $\beta > -1/(2H) + \lambda_1 + \lambda_2$. There exists a constant $C > 0$ such that for any $f \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\beta$, any $0 \le s \le u \le t \le 1$, any $\mathcal{F}_s$-measurable random variables $\kappa_1, \kappa_2 \in L^m$ and any $\mathcal{F}_u$-measurable random variables $\kappa_3, \kappa_4 \in L^m$, there is*

$$\begin{aligned} &\Big\| \int_u^t (f(B_r + \kappa_1) - f(B_r + \kappa_2) - f(B_r + \kappa_3) + f(B_r + \kappa_4))\, dr \Big\|_{L^m} \\ &\quad \le C \|f\|_{\mathcal{B}_p^\beta} \|\mathbb{E}^s[|\kappa_1 - \kappa_3|^m]^{1/m}\|_{L^\infty}^{\lambda_2} \|\kappa_1 - \kappa_2\|_{L^m}^{\lambda_1} (t-u)^{1+H(\beta - \lambda_1 - \lambda_2 - \frac{d}{p})} \\ &\quad\quad + C\|f\|_{\mathcal{B}_p^\beta} \|\kappa_1 - \kappa_2 - \kappa_3 + \kappa_4\|_{L^m}^{\lambda} (t-u)^{1+H(\beta - \lambda - \frac{d}{p})}. \end{aligned}$$

*Proof.* The proof is identical to the one-dimensional version of this result, see Corollary D.4 of [1], so we do not repeat it. It relies on Lemma 3.4 and Lemma 3.1(*iii*). □

**Proposition 5.2.** *Let $(\psi_t)_{t\in[0,1]}$, $(\phi_t)_{t\in[0,1]}$ be two $\mathbb{R}^d$-valued stochastic processes adapted to $\mathbb{F}$. Let $f \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\gamma$ and $m \in [2, \infty)$ such that $m \le p$. Assume that $\gamma - \frac{d}{p} > 1 - \frac{1}{2H}$ and let $\tau \in (0, 1)$ such that*

$$\left(\tau \wedge \frac{1}{2}\right) + H\left(\gamma - 1 - \frac{d}{p}\right) > 0. \tag{5.1}$$

*There exists a constant $C := C(m, p, \gamma, d) > 0$ such that for any $0 \le S < T \le 1$ and $(s, t) \in \Delta_{S,T}$,*

$$\begin{aligned} &\Big\| \int_s^t f(\psi_r + B_r) - f(\phi_r + B_r)\, dr \Big\|_{L^m} \\ &\quad \le C \|f\|_{\mathcal{B}_p^\gamma} \Big(1 + [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}}\Big) \Big([\psi - \phi]_{\mathcal{C}^\tau_{[S,T]} L^m} + \|\psi_S - \phi_S\|_{L^m}\Big) (t-s)^{1+H(\gamma - 1 - \frac{d}{p})}. \end{aligned} \tag{5.2}$$

*Proof.* Let $0 \le S < T \le 1$. For $(s, t) \in \Delta_{S,T}$, let

$$A_{s,t} = \int_s^t f(\psi_s + B_r) - f(\phi_s + B_r)\, dr \quad \text{and} \quad \mathcal{A}_t = \int_S^t f(\psi_r + B_r) - f(\phi_r + B_r)\, dr. \tag{5.3}$$

Assume without any loss of generality that $[\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}}$ and $[\psi - \phi]_{\mathcal{C}^\tau_{[S,T]} L^m}$ are finite, otherwise the result is trivial.

Let $\varepsilon \in (0, \gamma - (1 - 1/(2H)))$. In the following, we check the conditions in order to apply Lemma A.1 (with $q = m$). To show that (A.2) and (A.3) hold true with $\varepsilon_1 = \tau \wedge \frac{1}{2} + H(\gamma - 1 - d/p) > 0$, $\alpha_1 = 0$ and $\varepsilon_2 = 1/2 + H(\gamma - 1 - d/p) + \varepsilon/2 > 0$, $\alpha_2 = 0$, we prove that there exists a constant $C > 0$ independent of $s, t, S$ and $T$ such that for $u = (s + t)/2$,

(i) $\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \le C \|f\|_{\mathcal{B}_p^\gamma} ([\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}} + 1)([\psi - \phi]_{\mathcal{C}^\tau_{[S,T]} L^m} + \|\psi_S - \phi_S\|_{L^m})(t-s)^{1+\varepsilon_1}$;

(ii) $\|\delta A_{s,u,t}\|_{L^m} \le C \|f\|_{\mathcal{B}_p^\gamma} \Big([\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}} + 1\Big)([\psi - \phi]_{\mathcal{C}^\tau_{[S,T]} L^m} + \|\psi_S - \phi_S\|_{L^m}) (t-u)^{\frac{1}{2}+\varepsilon_2}$;

(iii) If (i) and (ii) are satisfied, (A.4) gives the convergence in probability of $\sum_{i=1}^{N_k - 1} A_{t_i^k, t_{i+1}^k}$ along any sequence of partitions $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$ of $[S, t]$ with mesh converging to 0. We will prove that the limit is the process $\mathcal{A}$ given in (5.3).

Assume for now that (i), (ii) and (iii) hold. Applying Lemma A.1 and recalling (A.1), we obtain that

$$\begin{aligned} &\Big\| \int_s^t f(B_r + \psi_r) - f(B_r + \phi_r)\, dr \Big\|_{L^m} \\ &\quad \le \|A_{s,t}\|_{L^m} + C \|f\|_{\mathcal{B}_p^\gamma} ([\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}} + 1)([\psi - \phi]_{\mathcal{C}^\tau_{[S,T]} L^m} + \|\psi_S - \phi_S\|_{L^m})(t-s)^{1+H(\gamma - 1 - \frac{d}{p}) + \tau\wedge\frac{1}{2}} \\ &\quad\quad + C \|f\|_{\mathcal{B}_p^\gamma} ([\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}} + 1)([\psi - \phi]_{\mathcal{C}^\tau_{[S,T]} L^m} + \|\psi_S - \phi_S\|_{L^m})(t-s)^{1+H(\gamma - 1 - \frac{d}{p}) + \frac{\varepsilon}{2}}. \end{aligned} \tag{5.4}$$

To bound $\|A_{s,t}\|_{L^m}$, we apply Lemma 3.4 with $q = m$ and $\beta = \gamma - 1$, and for $\Xi = (\psi_s, \phi_s)$. As $f$ is smooth and bounded, the first assumption of Lemma 3.4 is verified. By Lemma 3.1$(i)$, $\|f(\cdot + \psi_s) - f(\cdot + \phi_s)\|_{\mathcal{B}_p^{\gamma-1}} \leq 2\|f\|_{\mathcal{B}_p^{\gamma-1}}$, hence the second assumption of Lemma 3.4 is verified. It follows by Lemma 3.4 that

$$
\begin{aligned}
\|A_{s,t}\|_{L^m} &\leq C \, \| \|f(\psi_s + \cdot) - f(\phi_s + \cdot)\|_{\mathcal{B}_p^{\gamma-1}} \|_{L^m} \, (t-s)^{1+H(\gamma-1-\frac{d}{p})} \\
&\leq C \, \|f\|_{\mathcal{B}_p^{\gamma}} \, \|\psi_s - \phi_s\|_{L^m} \, (t-s)^{1+H(\gamma-1-\frac{d}{p})} \\
&\leq C \, \|f\|_{\mathcal{B}_p^{\gamma}} ([\psi - \phi]_{\mathcal{C}^{\tau}_{[S,T]} L^m} + \|\psi_S - \phi_S\|_{L^m}) \, (t-s)^{1+H(\gamma-1-\frac{d}{p})}. \qquad (5.5)
\end{aligned}
$$

Injecting the previous bound in (5.4), we get (5.2).

We now check that the conditions (i), (ii) and (iii) actually hold.

Proof of (i): For $u \in [s,t]$, we have $\delta A_{s,u,t} = \int_u^t f(\psi_s + B_r) - f(\phi_s + B_r) - f(\psi_u + B_r) + f(\phi_u + B_r) \, dr$. By the tower property of conditional expectation and Fubini's theorem, we have

$$
\begin{aligned}
\mathbb{E}^s \delta A_{s,u,t} &= \int_u^t \mathbb{E}^s \, \mathbb{E}^u \Big[ f(\psi_s + B_r) - f(\phi_s + B_r) - f(\psi_u + B_r) + f(\phi_u + B_r) \Big] \, dr \\
&=: \int_u^t \mathbb{E}^s \, \mathbb{E}^u [F(B_r, s, u) + \tilde{F}(B_r, s, u)] \, dr, \qquad (5.6)
\end{aligned}
$$

where

$$
\begin{aligned}
F(\cdot, s, u) &= f(\psi_s + \cdot) - f(\phi_s + \cdot) - f(\psi_u + \cdot) + f(\psi_u + \phi_s - \psi_s + \cdot), \\
\tilde{F}(\cdot, s, u) &= f(\phi_u + \cdot) - f(\psi_u + \phi_s - \psi_s + \cdot).
\end{aligned}
$$

By Lemma 3.3$(ii)$, we have that

$$
\begin{aligned}
|\mathbb{E}^u F(B_r, s, u)| &\leq \|F(\cdot, s, u)\|_{\mathcal{B}_p^{\gamma-2}} \, (r-u)^{H(\gamma-2-\frac{d}{p})}, \\
|\mathbb{E}^u \tilde{F}(B_r, s, u)| &\leq \|\tilde{F}(\cdot, s, u)\|_{\mathcal{B}_p^{\gamma-1}} \, (r-u)^{H(\gamma-1-\frac{d}{p})}.
\end{aligned}
$$

Moreover, by Lemma 3.1$(iii)$, it comes that

$$
\begin{aligned}
\mathbb{E}^s \|F(\cdot, s, u)\|_{\mathcal{B}_p^{\gamma-2}} &\leq \|f\|_{\mathcal{B}_p^{\gamma}} \, |\psi_s - \phi_s| \, \mathbb{E}^s |\psi_s - \psi_u| \\
&\leq \|f\|_{\mathcal{B}_p^{\gamma}} \, |\psi_s - \phi_s| \, (\mathbb{E}^s |\psi_s - \psi_u|^m)^{\frac{1}{m}} \\
&\leq \|f\|_{\mathcal{B}_p^{\gamma}} \, |\psi_s - \phi_s| \, [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}} (u-s)^{\frac{1}{2}+H}.
\end{aligned}
$$

Besides

$$
\begin{aligned}
\| \|\tilde{F}(\cdot, s, u)\|_{\mathcal{B}_p^{\gamma-1}} \|_{L^m} &\leq \|f\|_{\mathcal{B}_p^{\gamma}} \|\psi_s - \psi_u - \phi_s + \phi_u\|_{L^m} \\
&\leq \|f\|_{\mathcal{B}_p^{\gamma}} \, [\psi - \phi]_{\mathcal{C}^{\tau}_{[S,T]} L^m} \, (u-s)^{\tau}.
\end{aligned}
$$

Plugging the previous bounds in (5.6) and using $\|\psi_s - \phi_s\|_{L^m} \leq \|\psi_S - \phi_S\|_{L^m} + (T-S)^{\tau} [\psi - \phi]_{\mathcal{C}^{\tau}_{[S,T]} L^m}$, we obtain

$$
\begin{aligned}
\|\mathbb{E}^s \delta A_{s,u,t}\|_{L^m} &\leq C \, \|f\|_{\mathcal{B}_p^{\gamma}} \Big( [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}} + 1 \Big) ([\psi - \phi]_{\mathcal{C}^{\tau}_{[S,T]} L^m} + \|\psi_S - \phi_S\|_{L^m}) \\
&\quad \times \Big( (t-s)^{1+\tau+H(\gamma-1-\frac{d}{p})} + (t-s)^{1+\frac{1}{2}+H+H(\gamma-2-\frac{d}{p})} \Big) \\
&\leq C \, \|f\|_{\mathcal{B}_p^{\gamma}} \Big( [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}} + 1 \Big) ([\psi - \phi]_{\mathcal{C}^{\tau}_{[S,T]} L^m} + \|\psi_S - \phi_S\|_{L^m}) (t-s)^{1+\tau \wedge \frac{1}{2}+H(\gamma-1-\frac{d}{p})}.
\end{aligned}
$$

Proof of (ii): Apply Corollary 5.1 with $\beta = \gamma$, $\lambda = 1$, $\lambda_1 = 1$, $\lambda_2 = \varepsilon$, $\kappa_1 = \psi_s$, $\kappa_2 = \phi_s$, $\kappa_3 = \psi_u$ and $\kappa_4 = \phi_u$. This yields

$$
\begin{aligned}
\|\delta A_{s,u,t}\|_{L^m} &\leq C \|f\|_{\mathcal{B}_p^{\gamma}} \, \|\mathbb{E}^s [|\psi_s - \psi_u|^m]^{1/m}\|_{L^\infty}^{\varepsilon} \, \|\psi_s - \phi_s\|_{L^m} \, (t-u)^{1+H(\gamma-1-\varepsilon-d/p)} \\
&\quad + C \|f\|_{\mathcal{B}_p^{\gamma}} \, \|\psi_s - \phi_s - \psi_u + \phi_u\|_{L^m} \, (t-u)^{1+H(\gamma-1-d/p)}.
\end{aligned}
$$

Hence we get from (2.1) that

$$\|\delta A_{s,u,t}\|_{L^m} \leq C\|f\|_{\mathcal{B}_p^\gamma} [\psi]^{\varepsilon}_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}} \|\psi_s - \phi_s\|_{L^m} (t-u)^{1+H(\gamma-1-\frac{d}{p})+\frac{\varepsilon}{2}}$$
$$+ C\|f\|_{\mathcal{B}_p^\gamma} [\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]} L^m} (t-u)^{1+H(\gamma-1-\frac{d}{p})+\tau},$$

and use $[\psi]^{\varepsilon}_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}} \leq [\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}} + 1$ and $\|\psi_s - \phi_s\|_{L^m} \leq \|\psi_S - \phi_S\|_{L^m} + [\psi-\phi]_{\mathcal{C}^{\tau}_{[S,T]} L^m}$ to prove (ii).

Proof of (iii): Finally, for a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[S,t]$ with $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$ and mesh size $|\Pi_k|$ converging to zero, we have

$$\begin{aligned}\Big\|\mathcal{A}_t - \sum_{i=1}^{N_k-1} A_{t_i^k, t_{i+1}^k}\Big\|_{L^m} &\leq \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \Big\|f(\psi_r+B_r) - f(\phi_r+B_r) - f(\psi_{t_i^k}+B_r) + f(\phi_{t_i^k}+B_r)\Big\|_{L^m} dr\\ &\leq \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \Big\|f(\psi_r+B_r) - f(\psi_{t_i^k}+B_r)\Big\|_{L^m} + \Big\|f(\phi_r+B_r) - f(\phi_{t_i^k}+B_r)\Big\|_{L^m} dr\\ &\leq \|f\|_{\mathcal{C}^1} \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \Big\|\psi_r - \psi_{t_i^k}\Big\|_{L^m} + \Big\|\phi_r - \phi_{t_i^k}\Big\|_{L^m} dr.\end{aligned}$$

Now use that $\|\psi_r - \psi_{t_i^k}\|_{L^m} \leq [\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^m} |\Pi_k|^{1/2+H} \leq [\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}} |\Pi_k|^{1/2+H}$ and for $\sigma = \tau \wedge (1/2+H)$, $\|\phi_r - \phi_{t_i^k}\|_{L^m} \leq C([\phi-\psi]_{\mathcal{C}^{\tau}_{[S,T]} L^m} + [\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}})|\Pi_k|^{\sigma}$ to get

$$\Big\|\mathcal{A}_t - \sum_{i=1}^{N_k-1} A_{t_i^k, t_{i+1}^k}\Big\|_{L^m} \leq C\|f\|_{\mathcal{C}^1}(T-S)([\phi-\psi]_{\mathcal{C}^{\tau}_{[S,T]} L^m} + [\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}})|\Pi_k|^{\sigma} \underset{k\to\infty}{\longrightarrow} 0.$$

□

We now apply Proposition 5.2 in the sub-critical case to obtain the following bound on $E^{1,h,n}$. This bound is used in Section 4.

**Corollary 5.3.** *Recall that the process $K^{h,n}$ was defined in* (4.1) *and let $X_0$ be an $\mathcal{F}_0$-measurable random variable. Let $m \in [2,\infty)$ such that $m \leq p$ and assume that $\gamma - d/p > 1 - 1/(2H)$. There exists a constant $C > 0$ such that for any $0 \leq S < T \leq 1$, any $(s,t) \in \Delta_{S,T}$, any $h \in (0,1)$ and any $n \in \mathbb{N}$,*

$$\begin{aligned}&\Big\|\int_s^t b^n(X_0+K_r+B_r) - b^n(X_0+K_r^{h,n}+B_r)\,dr\Big\|_{L^m}\\ &\quad \leq C\|b\|_{\mathcal{B}_p^\gamma}\Big(1+[X-B]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}}\Big)\Big([K-K^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]} L^m} + \|K_S - K_S^{h,n}\|_{L^m}\Big)(t-s)^{1+H(\gamma-1-\frac{d}{p})}.\end{aligned}$$

*Proof.* Notice that $\tau = 1/2$ satisfies (5.1). Hence apply Proposition 5.2 with $\tau = 1/2$, $f = b^n$, $\psi = X_0 + K$ and $\phi = X_0 + K^{h,n}$ and recall from (2.4) that $\|b^n\|_{\mathcal{B}_p^\gamma} \leq \|b\|_{\mathcal{B}_p^\gamma}$ to get the result. □

The following proposition provides a result similar to Corollary 5.3 but in the critical case.

**Proposition 5.4.** *Let the assumptions of Theorem 2.5(c) hold. In particular, recall that $X_0$ is an $\mathcal{F}_0$-measurable random variable, that $\gamma - d/p = 1 - 1/(2H)$ and $\gamma > 1 - 1/(2H)$, $\zeta \in (0,1/2)$ and assume further that $2 \leq m \leq p$. Recall also that $K^n$ and $K^{h,n}$ were defined in* (4.1)*, and $\epsilon(h,n)$ was defined in* (4.4).

*There exist constants $\mathbf{M} > 0$ and $\ell_0 > 0$ such that for any $0 \le S < T \le 1$ which satisfy $T - S \le \ell_0$, any $(s,t) \in \Delta_{S,T}$, any $h \in (0,1)$ and any $n \in \mathbb{N}$,*

$$\begin{aligned}
&\Big\| \int_s^t b^n(X_0 + K_r + B_r) - b^n(X_0 + K_r^{h,n} + B_r)\, dr \Big\|_{L^m} \\
&\le \mathbf{M} \left( 1 + \Big| \log \frac{T^H(1 + [K^{h,n}]_{\mathcal{C}^{1/2+H}_{[S,T]} L^m})}{\|K - K^{h,n}\|_{L^\infty_{[S,T]} L^m} + \epsilon(h,n)} \Big| \right) \Big( \|K - K^{h,n}\|_{L^\infty_{[S,T]} L^m} + \epsilon(h,n) \Big) (t-s) \\
&\quad + \mathbf{M} \Big( \|K - K^{h,n}\|_{L^\infty_{[S,T]} L^m} + [K - K^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]} L^m} \Big) (t-s)^{\frac{1}{2}}.
\end{aligned}$$

**Remark 5.5.** *The constant $\mathbf{M}$ is important in the proof of Theorem 2.5(c) as it appears in the order of convergence.*

*Proof.* Let $0 \le S < T \le 1$. For $(s,t) \in \Delta_{S,T}$, let $A_{s,t}$ and $\mathcal{A}_t$ be defined by

$$\begin{aligned}
A_{s,t} &= \int_s^t b^n(X_0 + K_s + B_r) - b^n(X_0 + K_s^{h,n} + B_r)\, dr, \\
\mathcal{A}_t &= \int_S^t b^n(X_0 + K_r + B_r) - b^n(X_0 + K_r^{h,n} + B_r)\, dr,
\end{aligned} \tag{5.7}$$

and let

$$R_{s,t} = \mathcal{A}_t - \mathcal{A}_s - A_{s,t}. \tag{5.8}$$

In this proof, we write $\|K - K^{h,n}\|_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]} L^m}$ for $\|K_S - K_S^{h,n}\|_{L^m} + [K - K^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]} L^m}$.

Let

$$0 < \varepsilon < \left( \gamma - 1 + \frac{1}{2H} \right) \wedge (1 - 2\zeta).$$

Set $\tau = 1/2 + \varepsilon/2$. In the following, we will check the conditions (A.2), (A.3) and (A.5) which permit to apply the stochastic sewing with critical exponent [3, Theorem 4.5]. To show that (A.2), (A.3) and (A.5) hold true with $\varepsilon_1 = H$, $\alpha_1 = 0$, $\varepsilon_2 = 1/2 + H(\gamma - 1 - d/p) + \varepsilon/2 = \varepsilon/2$, $\alpha_2 = 0$ and $\varepsilon_4 = \varepsilon/2$ we prove that there exists a constant $C > 0$ independent of $s, t, S$ and $T$ such that for $u = (s+t)/2$,

- (i) $\|\mathbb{E}^s \delta A_{s,u,t}\|_{L^m} \le C\, (1 + [K^{h,n}]_{\mathcal{C}^{1/2+H}_{[S,T]} L^m})\, (t-s)^{1+H}$;
- (i') $\|\mathbb{E}^s \delta A_{s,u,t}\|_{L^m} \le C\, [R]_{\mathcal{C}^{\tau}_{[S,T]} L^m}\, (t-s)^{\frac{1}{2}+\tau} + C\Big( \|K - K^{h,n}\|_{L^\infty_{[S,T]} L^m} + \epsilon(h,n) \Big)(t-s)$;
- (ii) $\|\delta A_{s,u,t}\|_{L^m} \le C\, \|K - K^{h,n}\|_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]} L^m}\, (t-s)^{\frac{1}{2}+\varepsilon_2}$;
- (iii) If (i) and (ii) are satisfied, (A.4) gives the convergence in probability of $\sum_{i=1}^{N_k - 1} A_{t_i^k, t_{i+1}^k}$ along any sequence of partitions $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$ of $[S,t]$ with mesh converging to 0. We will prove that the limit is the process $\mathcal{A}$ given in (5.7).

Then from [3, Theorem 4.5], we get

$$\begin{aligned}
\|R_{s,t}\|_{L^m} &\le C \left( 1 + \Big| \log \frac{T^H(1 + [K^{h,n}]_{\mathcal{C}^{1/2+H}_{[S,T]} L^m})}{\|K - K^{h,n}\|_{L^\infty_{[S,T]} L^m} + \epsilon(h,n)} \Big| \right) \Big( \|K - K^{h,n}\|_{L^\infty_{[S,T]} L^m} + \epsilon(h,n) \Big) (t-s) \\
&\quad + C\, \|K - K^{h,n}\|_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]} L^m}\, (t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}} + C\, [R]_{\mathcal{C}^{\tau}_{[S,T]} L^m}\, (t-s)^{\frac{1}{2}+\tau}.
\end{aligned}$$

Now recalling that $\tau = 1/2 + \varepsilon/2$, we can divide both sides by $(t-s)^\tau$ and take the supremum over $(s,t) \in \Delta_{S,T}$ to get

$$[R]_{\mathcal{C}^\tau_{[S,T]}L^m} \le C\left(1 + \Big|\log \frac{T^H(1+[K^{h,n}]_{\mathcal{C}^{1/2+H}_{[S,T]}L^m})}{\|K-K^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)}\Big|\right)\left(\|K-K^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)\right)(T-S)^{1-\tau}$$
$$+ C\,\|K-K^{h,n}\|_{\mathcal{C}^{\frac12-\zeta}_{[S,T]}L^m} + C\,[R]_{\mathcal{C}^\tau_{[S,T]}L^m}\,(T-S)^{\frac12}.$$

For $S<T$ such that $T-S \le (2C)^{-2} =: \ell_0$, we get that $C\,[R]_{\mathcal{C}^\tau_{[S,T]}L^m}\,(T-S)^{\frac12} \le (1/2)[R]_{\mathcal{C}^\tau_{[S,T]}L^m}$. We then subtract this quantity on both sides to get

$$[R]_{\mathcal{C}^\tau_{[S,T]}L^m} \le 2C\left(1 + \Big|\log \frac{T^H(1+[K^{h,n}]_{\mathcal{C}^{1/2+H}_{[S,T]}L^m})}{\|K-K^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)}\Big|\right)\left(\|K-K^{h,n}\|_{L^\infty_{[S,T]}L^m}+\epsilon(h,n)\right)(T-S)^{1-\tau}$$
$$+ 2C\,\|K-K^{h,n}\|_{\mathcal{C}^{\frac12-\zeta}_{[S,T]}L^m}.$$

We conclude using (5.13) and

$$\|\mathcal{A}_t - \mathcal{A}_s\|_{L^m} \le \|R_{s,t}\|_{L^m} + \|A_{s,t}\|_{L^m}.$$

We now check that the conditions (i), (i'), (ii) and (iii) actually hold.

Proof of (i): For $u \in [s,t]$, by the tower property of conditional expectation and Fubini's theorem, we have

$$\begin{aligned}\mathbb{E}^s \delta A_{s,u,t} &= \int_u^t \mathbb{E}^s\,\mathbb{E}^u\Big[b^n(K_s+B_r) - b^n(K^{h,n}_s+B_r) - b^n(K_u+B_r) + b^n(K^{h,n}_u+B_r)\Big]\,dr \\ &=: \int_u^t \mathbb{E}^s\,\mathbb{E}^u[F(B_r,s,u) + \tilde F(B_r,s,u)]\,dr, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} F(\cdot,s,u) &= b^n(X_0+K_s+B_r) - b^n(X_0+K^{h,n}_s+B_r) - b^n(X_0+K_u+B_r) \\ &\quad + b^n(X_0+K^{h,n}_s+K_u-K_s+\cdot), \\ \tilde F(\cdot,s,u) &= b^n(X_0+K^{h,n}_u+\cdot) - b^n(X_0+K^{h,n}_s+K_u-K_s+\cdot). \end{aligned}$$

By Lemma 3.3$(ii)$, we have for $\lambda \in [0,1]$ that

$$|\mathbb{E}^u F(B_r,s,u)| \le C\,\|F(\cdot,s,u)\|_{\mathcal{B}^{\gamma-1-\lambda}_p}\,(r-u)^{H(\gamma-1-\lambda-\frac{d}{p})}.$$

Moreover, by Lemma 3.1$(iii)$, it comes that

$$\|F(\cdot,s,u)\|_{\mathcal{B}^{\gamma-1-\lambda}_p} \le C\,\|b^n\|_{\mathcal{B}^\gamma_p}|K_s-K_u|\,|K_s-K^{h,n}_s|^\lambda.$$

Hence

$$\begin{aligned}|\mathbb{E}^s\mathbb{E}^u F(B_r,s,u)| &\le C\,\|b^n\|_{\mathcal{B}^\gamma_p}\,|K_s-K^{h,n}_s|^\lambda\,\mathbb{E}^s|K_s-K_u|\,(r-u)^{H(\gamma-1-\lambda-\frac{d}{p})} \\ &\le C\,\|b^n\|_{\mathcal{B}^\gamma_p}\,|K_s-K^{h,n}_s|^\lambda\,[K]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\,(u-s)^{\frac12+H}\,(r-u)^{H(\gamma-1-\lambda-\frac{d}{p})}\end{aligned}$$

and from Jensen's inequality,

$$\|\mathbb{E}^s\mathbb{E}^u F(B_r,s,u)\|_{L^m} \le C\,\|b^n\|_{\mathcal{B}^\gamma_p}\,\|K_s-K^{h,n}_s\|^\lambda_{L^m}\,[K]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\,(u-s)^{\frac12+H}\,(r-u)^{H(\gamma-1-\lambda-\frac{d}{p})}. \tag{5.10}$$

As for $\tilde{F}$, we have similarly that

$$\begin{aligned}\|\mathbb{E}^s\mathbb{E}^u\tilde{F}(B_r,s,u)\|_{L^m} &\le C\,\|\mathbb{E}^s\|\tilde{F}(\cdot,s,u)\|_{\mathcal{B}_p^{\gamma-1}}\|_{L^m}\,(r-u)^{H(\gamma-1-\frac{d}{p})}\\ &\le C\,\|b^n\|_{\mathcal{B}_p^{\gamma}}\,\|K_s^{h,n}-K_u^{h,n}-K_s+K_u\|_{L^m}\,(r-u)^{H(\gamma-1-\frac{d}{p})}\\ &\le C\,\|b^n\|_{\mathcal{B}_p^{\gamma}}\left([K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}+[K]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}\right)(u-s)^{\frac12+H}\,(r-u)^{H(\gamma-1-\frac{d}{p})}.\end{aligned} \tag{5.11}$$

Choosing $\lambda=0$ and noticing that $H(\gamma-1-d/p)=-1/2$, we plug (5.10) and (5.11) in (5.9) to obtain

$$\begin{aligned}\|\mathbb{E}^s\delta A_{s,u,t}\|_{L^m} &\le C\,\|b^n\|_{\mathcal{B}_p^{\gamma}}\,[K]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\,(u-s)\,(t-u)^{\frac12}\\ &\quad + C\,\|b^n\|_{\mathcal{B}_p^{\gamma}}\left([K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}+[K]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}\right)(u-s)^{\frac12+H}\,(r-u)^{\frac12}\\ &\le C\,\|b^n\|_{\mathcal{B}_p^{\gamma}}\left([K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}+[K]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}\right)(t-s)^{1+H}.\end{aligned}$$

Use finally the property $K=X-B\in\mathcal{C}^{1/2+H}_{[0,1]}L^{m,\infty}$ from Theorem 2.3 and (2.4) to deduce (i).

Proof of (i'): We rely again on the decomposition (5.9).

We now use (5.10) with $\lambda=0$ and (5.11). Since $H(\gamma-2-d/p)=-H-1/2>-1$, there is

$$\begin{aligned}\|\mathbb{E}^s\delta A_{s,u,t}\|_{L^m} &\le C\,\|b^n\|_{\mathcal{B}_p^{\gamma}}\,[K]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\|K_s-K_s^{h,n}\|_{L^m}\,(t-s)\\ &\quad + C\,\|b^n\|_{\mathcal{B}_p^{\gamma}}\,\|K_u^{h,n}-K_s^{h,n}-K_u+K_s\|_{L^m}\,(t-s)^{\frac12}.\end{aligned} \tag{5.12}$$

Here we do not expect $K^{h,n}-K$ to be 1/2-Hölder continuous uniformly in $h$ and $n$, but only $(1/2-\zeta)$-Hölder, so we need to decompose $\|K_u^{h,n}-K_s^{h,n}-K_u+K_s\|_{L^m}$ into several terms. First we introduce the pivot term $K_u^n-K_s^n$ to get

$$\|K_u^{h,n}-K_s^{h,n}-K_u+K_s\|_{L^m}\le [K-K^n]_{\mathcal{C}^{\frac12}_{[S,T]}L^m}(u-s)^{\frac12}+\|K_u^{h,n}-K_s^{h,n}-K_u^n+K_s^n\|_{L^m}.$$

Now observe that from (4.1), (5.7) and (5.8),

$$R_{s,u}=K_u^n-K_s^n-A_{s,u}-\int_s^u b^n(X_0+K_r^{h,n}+B_r)\,dr,$$

so that

$$K_u^{h,n}-K_s^{h,n}-K_u^n+K_s^n=\int_s^u b^n(X_0+K_{r_h}^{h,n}+B_r)-b^n(X_0+K_r^{h,n}+B_r)\,dr-A_{s,u}-R_{s,u}.$$

Hence recalling the definition of $E^{2,h,n}$ from (4.3), we get

$$\|K_u^{h,n}-K_s^{h,n}-K_u^n+K_s^n\|_{L^m}\le\|E^{2,h,n}_{s,u}\|_{L^m}+\|A_{s,u}\|_{L^m}+\|R_{s,u}\|_{L^m}.$$

As in (5.5), we have

$$\begin{aligned}\|A_{s,u}\|_{L^m} &\le C\|b^n\|_{\mathcal{B}_p^{\gamma}}\,\|K_s-K_s^{h,n}\|_{L^m}\,(u-s)^{1+H(\gamma-1-\frac{d}{p})}\\ &= C\|b^n\|_{\mathcal{B}_p^{\gamma}}\,\|K_s-K_s^{h,n}\|_{L^m}\,(u-s)^{\frac12}.\end{aligned} \tag{5.13}$$

Thus we get

$$\begin{aligned}\|K_u^{h,n}-K_s^{h,n}-K_u^n+K_s^n\|_{L^m} &\le \left(C\|b^n\|_{\mathcal{B}_p^{\gamma}}\,\|K_s-K_s^{h,n}\|_{L^m}+[E^{2,h,n}]_{\mathcal{C}^{\frac12}_{[S,T]}L^m}\right)(u-s)^{\frac12}\\ &\quad +[R]_{\mathcal{C}^{\tau}_{[S,T]}L^m}\,(u-s)^{\tau}.\end{aligned}$$

Plugging the previous inequality in (5.12), it comes

$$\|\mathbb{E}^s\delta A_{s,u,t}\|_{L^m} \leq C\,\|b^n\|_{\mathcal{B}_p^\gamma}\Big((\|b^n\|_{\mathcal{B}_p^\gamma}+[K]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^{m,\infty}})\,\|K_s-K_s^{h,n}\|_{L^m}+[E^{2,h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m}\Big)(t-s)$$
$$+\,\|b^n\|_{\mathcal{B}_p^\gamma}\,[R]_{\mathcal{C}^\tau_{[S,T]}L^m}\,(t-s)^{\frac{1}{2}+\tau}.$$

Use finally that $[K]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^{m,\infty}}<\infty$, $[E^{2,h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m}\leq\epsilon(h,n)$ and (2.4) to deduce (i').

Proof of (ii): We apply Corollary 5.1 with $\beta=\gamma$, $\lambda=1$, $\lambda_1=1$, $\lambda_2=\varepsilon$, $\kappa_1=K_s$, $\kappa_2=K_s^{h,n}$, $\kappa_3=K_u$ and $\kappa_4=K_u^{h,n}$: this yields

$$\begin{aligned}\|\delta A_{s,u,t}\|_{L^m} &\leq C\|b^n\|_{\mathcal{B}_p^\gamma}\,\|\mathbb{E}^s[|K_s-K_u|^m]^{1/m}\|^\varepsilon_{L^\infty}\,\|K_s-K_s^{h,n}\|_{L^m}\,(t-u)^{1+H(\gamma-1-\varepsilon-d/p)}\\ &\quad+C\|b^n\|_{\mathcal{B}_p^\gamma}\,\|K_s-K_s^{h,n}-K_u+K_u^{h,n}\|_{L^m}\,(t-u)^{1+H(\gamma-1-d/p)}\\ &\leq C\|b^n\|_{\mathcal{B}_p^\gamma}\,[K]^\varepsilon_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^{m,\infty}}\,\|K_s-K_s^{h,n}\|_{L^m}\,(t-u)^{\frac{1}{2}+\frac{\varepsilon}{2}}\\ &\quad+C\|b^n\|_{\mathcal{B}_p^\gamma}\,[K-K^{h,n}]_{\mathcal{C}^{\frac{1}{2}-\zeta}_{[S,T]}L^m}\,(t-u)^{1-\zeta}.\end{aligned}$$

Since $\sup_n\|b^n\|_{\mathcal{B}_p^\gamma}$, $[K]_{\mathcal{C}^{1/2+H}_{[S,T]}L^{m,\infty}}$ are finite and $\varepsilon<1-2\zeta$, we have obtained (ii).

Proof of (iii): The proof is identical to point (iii) of Proposition 5.2. □

## 5.2 Sewing bounds for the $d$-dimensional discrete fBm

First, we obtain in Subsection 5.2.1 the $\mathcal{C}^{1/2+H}_{[0,1]}L^{m,\infty}$ regularity of the tamed Euler scheme (1.2) under (H2) and (H3), then we proceed to prove an upper bound on $[E^{2,h,n}]_{\mathcal{C}^{1/2}_{[0,1]}L^m}$ in Section 5.2.2.

### 5.2.1 Hölder regularity of the tamed Euler scheme

**Lemma 5.6.** *Recall that $\gamma$ and $p$ satisfy* (H2). *Let $m\in[2,\infty)$, $q\in[m,\infty]$. There exists a constant $C>0$ such that for any $0\leq S<T\leq 1$, any $\mathbb{R}^d$-valued $\mathcal{F}_S$-measurable random variable $\psi$, any $f\in\mathcal{C}_b^\infty(\mathbb{R}^d,\mathbb{R}^d)\cap\mathcal{B}_p^\gamma$, any $h>0$ and any $(s,t)\in\Delta_{S,T}$, we have*

$$\Big\|\Big(\mathbb{E}^S\Big|\int_s^t f(\psi+B_{r_h})\,dr\Big|^m\Big)^{\frac{1}{m}}\Big\|_{L^q}\leq C\,\Big(\|f\|_\infty\,h^{\frac{1}{2}-H}+\|f\|_{\mathcal{B}_p^\gamma}\Big)(t-s)^{\frac{1}{2}+H}.$$

*Proof.* We will check the conditions in order to apply Lemma A.1. For $(s,t)\in\Delta_{S,T}$, let

$$A_{s,t}=\mathbb{E}^s\int_s^t f(\psi+B_{r_h})\,dr \ \text{ and } \ \mathcal{A}_t=\int_S^t f(\psi+B_{r_h})\,dr.$$

Let $u\in[s,t]$ and notice that $\mathbb{E}^s\delta A_{s,u,t}=0$, so (A.2) holds with $\Gamma_1=0$. We will prove that (A.3) holds with $\alpha_2=0$ and

$$\Gamma_2=C\|f\|_\infty h^{\frac{1}{2}-H}+C\|f\|_{\mathcal{B}_p^\gamma}.$$

**The case $t-s\leq 2h$.** In this case we have

$$|A_{s,t}|\leq\|f\|_\infty(t-s)\leq C\|f\|_\infty h^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}+H}. \tag{5.14}$$

**The case $t-s>2h$.** Here we split $A_{s,t}$ in two

$$A_{s,t}=\mathbb{E}^s\int_s^{s+2h}f(\psi+B_{r_h})\,dr+\mathbb{E}^s\int_{s+2h}^t f(\psi+B_{r_h})\,dr.$$

For the first part, we obtain

$$\Big|\mathbb{E}^s \int_s^{s+2h} f(\psi + B_{r_h})\, dr\Big| \leq 2h\, \|f\|_\infty \leq C\, \|f\|_\infty h^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}+H}.$$

Denote the second part by

$$J := \int_{s+2h}^{t} \mathbb{E}^s f(\psi + B_{r_h})\, dr.$$

Using Lemma 3.3($ii$) and Lemma 3.1($i$), we have

$$\|J\|_{L^q} \leq C \int_{s+2h}^{t} \|f\|_{\mathcal{B}_p^\gamma} (r_h - s)^{H(\gamma-\frac{d}{p})}\, dr.$$

Since $2(r_h - s) \geq r - s$, we obtain

$$\begin{aligned} \|J\|_{L^q} &\leq C \int_{s+2h}^{t} \|f\|_{\mathcal{B}_p^\gamma} (r-s)^{H(\gamma-\frac{d}{p})}\, dr \\ &\leq C\|f\|_{\mathcal{B}_p^\gamma}(t-s)^{1+H(\gamma-\frac{d}{p})} \\ &\leq C\|f\|_{\mathcal{B}_p^\gamma}(t-s)^{\frac{1}{2}+H}. \end{aligned} \tag{5.15}$$

Overall, combining (5.14) and (5.15), we obtain that for all $s \leq t$,

$$\|A_{s,t}\|_{L^q} \leq C\Big(\|f\|_\infty h^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}+H} + \|f\|_{\mathcal{B}_p^\gamma}(t-s)^{\frac{1}{2}+H}\Big).$$

Thus for any $u \in [s,t]$,

$$\begin{aligned} \|\delta A_{s,u,t}\|_{L^q} &\leq \|A_{s,t}\|_{L^m} + \|A_{s,u}\|_{L^m} + \|A_{u,t}\|_{L^m} \\ &\leq C\Big(\|f\|_\infty h^{\frac{1}{2}-H} + \|f\|_{\mathcal{B}_p^\gamma}\Big)(t-s)^{\frac{1}{2}+H}. \end{aligned}$$

The power in $(t-s)$ is strictly larger than 1/2, so (A.3) holds.

**Convergence in probability.** Finally, for a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[S,t]$ with $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$ and mesh size converging to zero, we have

$$\begin{aligned} \Big\|\mathcal{A}_t - \sum_{i=1}^{N_k-1} A_{t_i^k, t_{i+1}^k}\Big\|_{L^1} &\leq \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \mathbb{E}\Big|f(\psi + B_{r_h}) - \mathbb{E}^{t_i^k}[f(\psi - B_{r_h})]\Big|\, dr \\ &\leq \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \mathbb{E}\Big|f(\psi + B_{r_h}) - \mathbb{E}^{t_i^k} f(\psi + B_{r_h})\Big|\, dr. \end{aligned}$$

Note that if $r_h \leq t_i^k$, then $\mathbb{E}|f(\psi+B_{r_h}) - \mathbb{E}^{t_i^k} f(\psi+B_{r_h})| = 0$. On the other hand, when $r_h \in (t_i^k, t_{i+1}^k]$ then in view of Lemma 3.3($iii$), we have

$$\mathbb{E}|f(\psi + B_{r_h}) - \mathbb{E}^{t_i^k} f(\psi + B_{r_h})| \leq C\|f\|_{\mathcal{C}^1}|\Pi_k|^H.$$

It follows that

$$\Big\|\mathcal{A}_t - \sum_{i=1}^{N_k-1} A_{t_i^k, t_{i+1}^k}\Big\|_{L^1} \leq \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \|f\|_{\mathcal{C}^1}|\Pi_k|^H\, dr,$$

and therefore $\sum_{i=1}^{N_k-1} A_{t_i^k, t_{i+1}^k}$ converges in probability to $\mathcal{A}_t$ as $k \to +\infty$. Hence we can apply Lemma A.1 with $\varepsilon_1 > 0$ and $\varepsilon_2 = H$ to conclude that

$$\begin{aligned} \|(\mathbb{E}^S|\mathcal{A}_t - \mathcal{A}_s|^m)^{\frac{1}{m}}\|_{L^q} &\leq \|(\mathbb{E}^S|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}|^m)^{\frac{1}{m}}\|_{L^q} + \|A_{s,t}\|_{L^q} \\ &\leq C\left(\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon} + \|f\|_{\mathcal{B}_p^\gamma}\right)(t-s)^{\frac{1}{2}+H}. \end{aligned}$$

□

**Proposition 5.7.** *Recall that $\gamma$ and $p$ satisfy* (H2). *Let $\varepsilon \in (0, \frac{1}{2})$ and $m \in [2, \infty)$. There exists a constant $C > 0$ such that for any $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process $(\psi_t)_{t\in[0,1]}$, any $f \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_p^\gamma$, any $0 \le S < T \le 1$ and any $(s,t) \in \Delta_{S,T}$, we have*

$$\Big\| \Big( \mathbb{E}^S \Big| \int_s^t f(\psi_r + B_{r_h})\, dr \Big|^m \Big)^{\frac{1}{m}} \Big\|_{L^\infty} \le C \left( \|f\|_\infty h^{\frac{1}{2}-H} + \|f\|_{\mathcal{B}_p^\gamma} \right) (t-s)^{\frac{1}{2}+H} + C\, [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}} \left( \|f\|_{\mathcal{B}_p^\gamma} + \|f\|_{\mathcal{C}^1} h^{\frac{1}{2}+H-\varepsilon} \right) (t-s)^{1+\varepsilon}. \quad (5.16)$$

**Remark 5.8.** *A direct consequence of this proposition is that for any $(s,t) \in \Delta_{0,1}$, we have*

$$\Big\| \Big( \mathbb{E}^s \Big| \int_s^t f(\psi_r + B_{r_h})\, dr \Big|^m \Big)^{\frac{1}{m}} \Big\|_{L^\infty} \le C \left( \|f\|_\infty h^{\frac{1}{2}-H} + \|f\|_{\mathcal{B}_p^\gamma} \right) (t-s)^{\frac{1}{2}+H} + C\, [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[s,t]} L^{m,\infty}} \left( \|f\|_{\mathcal{B}_p^\gamma} + \|f\|_{\mathcal{C}^1} h^{\frac{1}{2}+H-\varepsilon} \right) (t-s)^{1+\varepsilon}. \quad (5.17)$$

*Proof.* We will check the conditions in order to apply Lemma A.1 (with $q = \infty$). Let $0 \le S < T \le 1$. Assume that $[\psi]_{\mathcal{C}^{1/2+H}_{[S,T]} L^{m,\infty}} < \infty$, otherwise (5.16) trivially holds. For any $(s,t) \in \Delta_{S,T}$, define

$$A_{s,t} = \int_s^t f(\psi_s + B_{r_h})\, dr \quad \text{and} \quad \mathcal{A}_t = \int_S^t f(\psi_r + B_{r_h})\, dr.$$

To show that (A.2) and (A.3) hold true with $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = H > 0$ and $\alpha_1 = \alpha_2 = 0$, we prove that there exists a constant $C > 0$ independent of $s, t, S$ and $T$ such that for $u = (s+t)/2$,

(i) $\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^\infty} \le C\, [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}} \left( \|f\|_{\mathcal{B}_p^\gamma} + \|f\|_{\mathcal{C}^1} h^{\frac{1}{2}+H-\varepsilon} \right) (t-s)^{1+\varepsilon}$;

(ii) $\|(\mathbb{E}^S |\delta A_{s,u,t}|^m)^{\frac{1}{m}}\|_{L^\infty} \le C \left( \|f\|_\infty h^{\frac{1}{2}-H} + \|f\|_{\mathcal{B}_p^\gamma} \right) (t-s)^{\frac{1}{2}+H}$;

(iii) If (i) and (ii) are satisfied, (A.4) gives the convergence in probability of $\sum_{i=1}^{N_k-1} A_{t_i^k, t_{i+1}^k}$ along any sequence of partitions $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$ of $[S,t]$ with mesh converging to 0. We will prove that the limit is the process $\mathcal{A}$ given above.

Assume for now that (i), (ii) and (iii) hold. Applying Lemma A.1, we obtain that

$$\Big\| \Big( \mathbb{E}^S \Big| \int_s^t f(\psi_r + B_{r_h})\, dr \Big|^m \Big)^{\frac{1}{m}} \Big\|_{L^\infty} \le C \left( \|f\|_\infty h^{\frac{1}{2}-H} + \|f\|_{\mathcal{B}_p^\gamma} \right) (t-s)^{\frac{1}{2}+H} + C\, [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}} \left( \|f\|_{\mathcal{B}_p^\gamma} + \|f\|_{\mathcal{C}^1} h^{\frac{1}{2}+H-\varepsilon} \right) (t-s)^{1+\varepsilon} + \|(\mathbb{E}^S |A_{s,t}|^m)^{\frac{1}{m}}\|_{L^\infty}.$$

Applying Lemma 5.6 with $q = \infty$ and $\psi = \psi_s$ for the last term of the previous equation, we get (5.16).

We now check that the conditions (i), (ii) and (iii) actually hold.

Proof of (i): We have

$$\mathbb{E}^s \delta A_{s,u,t} = \int_u^t \mathbb{E}^s [f(\psi_s + B_{r_h}) - f(\psi_u + B_{r_h})]\, dr.$$

**The case $t - u \le 2h$.** In this case, using the Lipschitz norm of $f$, we have

$$|\mathbb{E}^s \delta A_{s,u,t}| \le \|f\|_{\mathcal{C}^1} \int_u^t \mathbb{E}^s |\psi_s - \psi_u|\, dr \le \|f\|_{\mathcal{C}^1} (t-u) \left( \mathbb{E}^s |\psi_s - \psi_u|^m \right)^{\frac{1}{m}}.$$

Thus using the inequality $(t-u)(u-s)^{1/2+H} \le C\, h^{1/2+H-\varepsilon} (t-u)^{1/2-H+\varepsilon} (u-s)^{1/2+H} \le C(t-s)^{1+\varepsilon} h^{\frac{1}{2}+H-\varepsilon}$, it comes

$$\|\mathbb{E}^s \delta A_{s,u,t}\|_{L^\infty} \le C\, \|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]} L^{m,\infty}} (t-s)^{1+\varepsilon} h^{\frac{1}{2}+H-\varepsilon}.$$

**The case $t-u>2h$.** We split the integral between $u$ and $u+2h$ and then between $u+2h$ and $t$ as follows:

$$\begin{aligned}\mathbb{E}^s\delta A_{s,u,t} &= \int_u^{u+2h} \mathbb{E}^s[f(\psi_s+B_{r_h})-f(\psi_u+B_{r_h})]\,dr + \int_{u+2h}^t \mathbb{E}^s[f(\psi_s+B_{r_h})-f(\psi_u+B_{r_h})]\,dr \\ &=: J_1+J_2.\end{aligned}$$

For $J_1$, we obtain as in the case $t-u\leq 2h$ that

$$\begin{aligned}\|J_1\|_{L^\infty} = \|\mathbb{E}^s\delta A_{s,u,u+2h}\|_{L^\infty} &\leq C\|f\|_{\mathcal{C}^1}[\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}(u+2h-s)^{1+\varepsilon}h^{\frac12+H-\varepsilon} \\ &\leq C\|f\|_{\mathcal{C}^1}[\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}(t-s)^{1+\varepsilon}h^{\frac12+H-\varepsilon}. \end{aligned} \tag{5.18}$$

As for $J_2$, the tower property of the conditional expectation yields

$$J_2 = \mathbb{E}^s\int_{u+2h}^t \mathbb{E}^u[f(\psi_s+B_{r_h})-f(\psi_u+B_{r_h})]\,dr.$$

Now use Lemma 3.3($ii$) and Lemma 3.1($ii$) to obtain

$$\begin{aligned}|J_2| &\leq C\int_{u+2h}^t \mathbb{E}^s\|f(\psi_s+\cdot)-f(\psi_u+\cdot)\|_{\mathcal{B}^{\gamma-1}_p}(r_h-u)^{H(\gamma-\frac{d}{p}-1)}\,dr \\ &\leq C\,\|f\|_{\mathcal{B}^\gamma_p}\,(\mathbb{E}^s|\psi_s-\psi_u|^m)^{\frac1m}\int_{u+2h}^t (r_h-u)^{H(\gamma-\frac{d}{p}-1)}\,dr\,.\end{aligned}$$

Using the fact that $2(r_h-u)\geq(r-u)$, it comes

$$\begin{aligned}\|J_2\|_{L^\infty} &\leq C\,[\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\|f\|_{\mathcal{B}^\gamma_p}\,(u-s)^{\frac12+H}\int_{u+2h}^t(r-u)^{H(\gamma-\frac{d}{p}-1)}\,dr \\ &\leq C\,[\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\|f\|_{\mathcal{B}^\gamma_p}\,(u-s)^{\frac12+H}\,(t-u)^{1+H(\gamma-\frac{d}{p}-1)} \\ &\leq C\,[\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\|f\|_{\mathcal{B}^\gamma_p}\,(t-s)^{1+H}.\end{aligned} \tag{5.19}$$

In view of the inequalities (5.18) and (5.19), we have proven (i).

Proof of (ii): We write

$$\|(\mathbb{E}^S|\delta A_{s,u,t}|^m)^{\frac1m}\|_{L^\infty} \leq \|(\mathbb{E}^S|\delta A_{s,t}|^m)^{\frac1m}\|_{L^\infty} + \|(\mathbb{E}^S|\delta A_{s,u}|^m)^{\frac1m}\|_{L^\infty} + \|(\mathbb{E}^S|\delta A_{u,t}|^m)^{\frac1m}\|_{L^\infty}$$

Applying Lemma 5.6 with $q=\infty$ for each term in the right-hand side of the previous inequality, respectively for $\psi=\psi_s$, $\psi_s$ again and $\psi_u$, we get (ii).

Proof of (iii): Finally, for a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[S,t]$ with $\Pi_k=\{t_i^k\}_{i=1}^{N_k}$ and mesh size $|\Pi_k|$ converging to zero, we have

$$\begin{aligned}\Big\|\mathcal{A}_t-\sum_{i=1}^{N_k-1}A_{t_i,t_{i+1}}\Big\|_{L^1} &\leq \sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\mathbb{E}|f(\psi_r+B_{r_h})-f(\psi_{t_i^k}+B_{r_h})|\,dr \\ &\leq C\sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\|f\|_{\mathcal{C}^1}\|\psi_r-\psi_{t_i^k}\|_{L^1}\,dr \\ &\leq C\|f\|_{\mathcal{C}^1}\sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}[\psi]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^m}|\Pi_k|^{\frac12+H}\,dr \underset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

□

**Corollary 5.9.** *Assume* (H2)*, let $\mathcal{D}$ be a sub-domain of* $(0,1)\times\mathbb{N}$ *satisfying* (H3) *and let* $m\in[2,\infty)$*. Recall also that* $K^{h,n}$ *was defined in* (4.1)*. Then*

$$\sup_{(h,n)\in\mathcal{D}}[K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[0,1]}L^{m,\infty}}<\infty.$$

*Proof.* Let $(h,n)\in\mathcal{D}$ and $\varepsilon\le H$. In view of Equation (5.17), we have for $f=b^n$ and $\psi=X_0+K^{h,n}$ that there exists a constant $C$ such that for any $h\in(0,1)$, $n\in\mathbb{N}$ and $(s,t)\in\Delta_{0,1}$,

$$\begin{aligned}\Big\|\Big(\mathbb{E}^s\Big|\int_s^t b^n(X_0+K_r^{h,n}+B_{r_h})\,dr\Big|^m\Big)^{\frac1m}\Big\|_{L^\infty}&\le C\left(\|b^n\|_\infty h^{\frac12-H}+\|b\|_{\mathcal{B}_p^\gamma}\right)(t-s)^{\frac12+H}\\&\quad+C\,[K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[s,t]}L^{m,\infty}}\left(\|b\|_{\mathcal{B}_p^\gamma}+\|b^n\|_{\mathcal{C}^1}h^{\frac12+H-\varepsilon}\right)(t-s)^{1+\varepsilon},\end{aligned}$$

where we used that $\|b^n\|_{\mathcal{B}_p^\gamma}\le\|b\|_{\mathcal{B}_p^\gamma}$. In particular, for $0\le S<T\le1$ and any $(s,t)\in\Delta_{S,T}$,

$$\begin{aligned}\Big\|\Big(\mathbb{E}^s\Big|\int_s^t b^n(X_0+K_r^{h,n}+B_{r_h})\,dr\Big|^m\Big)^{\frac1m}\Big\|_{L^\infty}&\le C\left(\|b^n\|_\infty h^{\frac12-H}+\|b\|_{\mathcal{B}_p^\gamma}\right)(t-s)^{\frac12+H}\\&\quad+C\,[K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\left(\|b\|_{\mathcal{B}_p^\gamma}+\|b^n\|_{\mathcal{C}^1}h^{\frac12+H-\varepsilon}\right)(t-s)^{1+\varepsilon}.\end{aligned}$$

Moreover, using that $|K_r^{h,n}-K_{r_h}^{h,n}|\le\|b^n\|_\infty h$, we have

$$\begin{aligned}\Big\|\Big(\mathbb{E}^s|K_t^{h,n}-K_s^{h,n}|^m\Big)^{\frac1m}\Big\|_{L^\infty}&\le\Big\|\Big(\mathbb{E}^s\Big|\int_s^t b^n(X_0+K_{r_h}^{h,n}+B_{r_h})-b^n(X_0+K_r^{h,n}+B_{r_h})\,dr\Big|^m\Big)^{\frac1m}\Big\|_{L^\infty}\\&\quad+\Big\|\Big(\mathbb{E}^s\Big|\int_s^t b^n(X_0+K_r^{h,n}+B_{r_h})\,dr\Big|^m\Big)^{\frac1m}\Big\|_{L^\infty}\\&\le C\,\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h\,(t-s)+\Big\|\Big(\mathbb{E}^s\Big|\int_s^t b^n(X_0+K_r^{h,n}+B_{r_h})\,dr\Big|^m\Big)^{\frac1m}\Big\|_{L^\infty}\\&\le C\left(\|b^n\|_\infty h^{\frac12-H}+\|b\|_{\mathcal{B}_p^\gamma}\right)(t-s)^{\frac12+H}\\&\quad+C\left([K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}(\|b\|_{\mathcal{B}_p^\gamma}+\|b^n\|_{\mathcal{C}^1}h^{\frac12+H-\varepsilon})+\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h\right)(t-s).\end{aligned}$$

Now using (H3) with $\eta=\varepsilon$ small enough, we get $\sup_{(h,n)\in\mathcal{D}}\|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h<\infty$ and

$$\Big\|\Big(\mathbb{E}^s|K_t^{h,n}-K_s^{h,n}|^m\Big)^{\frac1m}\Big\|_{L^\infty}\le C\,(t-s)^{\frac12+H}+C\left([K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}+1\right)(t-s).$$

Now divide by $(t-s)^{\frac12+H}$ and take the supremum over $[S,T]$ to get that

$$[K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\le C+C\,[K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\,(T-S)^{\frac12-H}.$$

Let $\ell=(\frac{1}{2C})^{\frac{1}{1/2-H}}$. Then for $T-S\le\ell$, we deduce

$$[K^{h,n}]_{\mathcal{C}^{\frac12+H}_{[S,T]}L^{m,\infty}}\le 2C.$$

Since $\ell$ does not depend on $h$ nor $n$, we get the Hölder regularity on the whole interval $[0,1]$. ☐

### 5.2.2 Hölder regularity of $E^{2,h,n}$

We start this subsection with general results on the regularisation property of the discrete-time fBm, that eventually lead to a bound on the term $E^{2,h,n}$ in Corollary 5.13.

**Lemma 5.10.** *Let $\varepsilon \in (0, \frac{1}{2})$ and $m \in [2, \infty)$. There exists a constant $C > 0$ such that for any $0 \leq S < T \leq 1$, any $\mathbb{R}^d$-valued $\mathcal{F}_S$-measurable random variable $\psi$, any $f \in \mathcal{C}_b^0(\mathbb{R}^d, \mathbb{R}^d)$, any $h > 0$ and any $(s,t) \in \Delta_{S,T}$, we have*

$$\Big\| \int_s^t f(\psi + B_r) - f(\psi + B_{r_h})\, dr \Big\|_{L^m} \leq C \, \|f\|_\infty \, h^{\frac{1}{2}-\varepsilon} (t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

*Proof.* We will check the conditions in order to apply Lemma A.1. For $(s,t) \in \Delta_{S,T}$, let

$$A_{s,t} = \mathbb{E}^s \int_s^t f(\psi + B_r) - f(\psi + B_{r_h})\, dr \quad \text{and} \quad \mathcal{A}_t = \int_S^t f(\psi + B_r) - f(\psi + B_{r_h})\, dr.$$

Let $u \in [s,t]$ and notice that $\mathbb{E}^s \delta A_{s,u,t} = 0$, so (A.2) holds with $\Gamma_1 = 0$. We will prove that (A.3) holds with $\alpha_2 = 0$ and

$$\Gamma_2 = C \|f\|_\infty h^{\frac{1}{2}-\varepsilon}.$$

**The case $t - s \leq 2h$.** In this case we have

$$|A_{s,t}| \leq \|f\|_\infty (t-s) \leq C \|f\|_\infty h^{\frac{1}{2}-\varepsilon} (t-s)^{\frac{1}{2}+\varepsilon}. \tag{5.20}$$

**The case $t - s > 2h$.** Here we split $A_{s,t}$ in two:

$$A_{s,t} = \mathbb{E}^s \int_s^{s+2h} f(\psi + B_r) - f(\psi + B_{r_h})\, dr + \mathbb{E}^s \int_{s+2h}^t f(\psi + B_r) - f(\psi + B_{r_h})\, dr.$$

For the first part, we obtain

$$\Big| \mathbb{E}^s \int_s^{s+2h} f(\psi + B_r) - f(\psi + B_{r_h})\, dr \Big| \leq 4h \, \|f\|_\infty \leq C \, \|f\|_\infty h^{\frac{1}{2}-\varepsilon} (t-s)^{\frac{1}{2}+\varepsilon}.$$

Denote the second part by

$$J := \int_{s+2h}^t \mathbb{E}^s [f(\psi + B_r) - f(\psi + B_{r_h})]\, dr.$$

From Lemma 3.3($i$) and (3.1), we have

$$\begin{aligned} J &= \int_{s+2h}^t \Big( G_{C_1 (r-s)^{2H}} f(\psi + \mathbb{E}^s B_r) - G_{C_1 (r_h - s)^{2H}} f(\psi + \mathbb{E}^s B_{r_h}) \Big)\, dr \qquad (5.21) \\ &= \int_{s+2h}^t \Big( G_{C_1 (r-s)^{2H}} f(\psi + \mathbb{E}^s B_r) - G_{C_1 (r_h - s)^{2H}} f(\psi + \mathbb{E}^s B_r) \Big)\, dr \\ &\quad + \int_{s+2h}^t \big( G_{C_1 (r_h-s)^{2H}} f(\psi + \mathbb{E}^s B_r) - G_{C_1 (r_h - s)^{2H}} f(\psi + \mathbb{E}^s B_{r_h}) \big)\, dr \\ &=: J_1 + J_2. \end{aligned}$$

For $J_1$, we apply [6, Proposition 3.7 (ii)] with $\beta = 0$, $\delta = 1$, $\alpha = 0$ to get

$$\|J_1\|_{L^m} \leq C \, \|f\|_\infty \int_{s+2h}^t ((r-s)^{2H} - (r_h - s)^{2H})(r_h - s)^{-2H}\, dr.$$

Now applying the inequalities $(r-s)^{2H} - (r_h - s)^{2H} \leq C(r - r_h)(r_h - s)^{2H-1}$ and $2(r_h - s) \geq (r-s)$, it comes

$$\begin{aligned} \|J_1\|_{L^m} &\leq C \, \|f\|_\infty \int_{s+2h}^t (r - r_h)(r-s)^{2H-1} (r-s)^{-2H}\, dr \\ &\leq C \, \|f\|_\infty h \int_{s+2h}^t (r-s)^{-1}\, dr \\ &\leq C \, \|f\|_\infty h \left( |\log(2h)| + |\log(t-s)| \right). \end{aligned}$$

Use again that $2h < t - s$ to get

$$\|J_1\|_{L^m} \leq C\,\|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

As for $J_2$, we have

$$\|J_2\|_{L^m} \leq \int_{s+2h}^t \|G_{C_1(r_h-s)^{2H}} f\|_{\mathcal{C}^1}\, \|\mathbb{E}^s B_r - \mathbb{E}^s B_{r_h}\|_{L^m}\, dr.$$

In view of [6, Proposition 3.7 (i)] applied with $\beta = 1$, $\alpha = 0$, [6, Proposition 3.6 (v)] and using again that $2(r_h - s) \geq (r - s)$, we get

$$\begin{aligned}
\|J_2\|_{L^m} &\leq C\,\|f\|_\infty \int_{s+2h}^t \|\mathbb{E}^s B_r - \mathbb{E}^s B_{r_h}\|_{L^m} (r_h - s)^{-H}\, dr \\
&\leq C\,\|f\|_\infty \int_{s+2h}^t (r - r_h)(r-s)^{H-1}(r-s)^{-H}\, dr \\
&\leq C\,\|f\|_\infty\, h\,(|\log(2h)| + |\log(t-s)|) \\
&\leq C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.
\end{aligned}$$

Combining the bounds on $J_1$ and $J_2$, we deduce that

$$\|J\|_{L^m} \leq C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

Hence for all $t - s > 2h$,

$$\|A_{s,t}\|_{L^m} \leq C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}. \tag{5.22}$$

Overall, combining (5.20) and (5.22), we obtain that for all $s \leq t$,

$$\|A_{s,t}\|_{L^m} \leq C\|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

Thus for any $u \in [s,t]$,

$$\begin{aligned}
\|\delta A_{s,u,t}\|_{L^m} &\leq \|A_{s,t}\|_{L^m} + \|A_{s,u}\|_{L^m} + \|A_{u,t}\|_{L^m} \\
&\leq C\|f\|_\infty h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.
\end{aligned}$$

The power in $(t - s)$ is strictly larger than $1/2$, so (A.3) holds.

**Convergence in probability.** Finally, for a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[S,t]$ with $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$ and mesh size converging to zero, we have

$$\begin{aligned}
\Big\|\mathcal{A}_t - \sum_{i=1}^{N_k-1} A_{t_i^k, t_{i+1}^k}\Big\|_{L^1} &\leq \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \mathbb{E}\Big| f(\psi + B_r) - f(\psi + B_{r_h}) - \mathbb{E}^{t_i^k}[f(\psi+B_r) + f(\psi - B_{r_h})]\Big|\, dr \\
&\leq \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \mathbb{E}\Big| f(\psi + B_r) - \mathbb{E}^{t_i^k} f(\psi + B_r)\Big|\, dr \\
&\quad + \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \mathbb{E}\Big| f(\psi + B_{r_h}) - \mathbb{E}^{t_i^k} f(\psi + B_{r_h})\Big|\, dr \\
&=: I_1 + I_2.
\end{aligned}$$

In view of Lemma 3.3$(iii)$, it comes that

$$I_1 \leq \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \|f\|_{\mathcal{C}^1} (r - t_i^k)^H\, dr \leq \|f\|_{\mathcal{C}^1} |\Pi_k|^H\,(t - S).$$

As for $I_2$, note that if $r_h \le t_i^k$, then $\mathbb{E}|f(\psi+B_{r_h}) - \mathbb{E}^{t_i^k} f(\psi+B_{r_h})|= 0$. On the other hand, when $r_h \in (t_i^k, t_{i+1}^k]$ then in view of Lemma 3.3(*iii*), we have

$$\mathbb{E}|f(\psi+B_{r_h}) - \mathbb{E}^{t_i^k} f(\psi+B_{r_h})|\le C\|f\|_{\mathcal{C}^1}|\Pi_k|^H.$$

It follows that

$$I_2 \le \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \|f\|_{\mathcal{C}^1}|\Pi_k|^H \, dr,$$

and therefore $\sum_{i=1}^{N_k-1} A_{t_i^k,t_{i+1}^k}$ converges in probability to $\mathcal{A}_t$ as $k\to+\infty$. We can therefore apply Lemma A.1 with $\varepsilon_1>0$ and $\varepsilon_2=\varepsilon/2$ to conclude that

$$\begin{aligned}\|\mathcal{A}_t-\mathcal{A}_s\|_{L^m} &\le \|\mathcal{A}_t-\mathcal{A}_s-A_{s,t}\|_{L^m}+\|A_{s,t}\|_{L^m}\\ &\le C\,\|f\|_\infty\, h^{\frac12-\varepsilon}\,(t-s)^{\frac12+\frac{\varepsilon}{2}}.\end{aligned}$$

□

**Proposition 5.11.** *Let $\varepsilon\in(0,\frac12)$ and $m\in[2,\infty)$. There exists a constant $C>0$ such that for any $\mathbb{R}^d$-valued stochastic process $(\psi_t)_{t\in[0,1]}$ adapted to $\mathbb{F}$, any $f\in\mathcal{C}_b^1(\mathbb{R}^d,\mathbb{R}^d)$, any $h\in(0,1)$ and any $(s,t)\in\Delta_{0,1}$, we have*

$$\Big\|\int_s^t f(\psi_r+B_r)-f(\psi_r+B_{r_h})\,dr\Big\|_{L^m}\le C\Big(\|f\|_\infty h^{\frac12-\varepsilon}(t-s)^{\frac12+\frac{\varepsilon}{2}}+\|f\|_{\mathcal{C}^1}[\psi]_{\mathcal{C}^1_{[0,1]}L^\infty}h^{1-\varepsilon}(t-s)^{1+\frac{\varepsilon}{2}}\Big). \tag{5.23}$$

*Proof.* Assume that $[\psi]_{\mathcal{C}^1_{[0,1]}L^\infty}<\infty$, otherwise (5.23) trivially holds. We will check the conditions in order to apply Lemma A.1 (with $q=m$). Let and $0\le S<T\le1$. For any $(s,t)\in\Delta_{S,T}$, define

$$A_{s,t}=\int_s^t f(\psi_s+B_r)-f(\psi_s+B_{r_h})\,dr \quad\text{and}\quad \mathcal{A}_t=\int_S^t f(\psi_r+B_r)-f(\psi_r+B_{r_h})\,dr.$$

To show that (A.2) and (A.3) hold true with $\varepsilon_1=\varepsilon_2=\varepsilon/2>0$ and $\alpha_1=\alpha_2=0$, we prove that there exists a constant $C>0$ independent of $s,t,S$ and $T$ such that for $u=(s+t)/2$,

(i) $\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m}\le C\,\|f\|_{\mathcal{C}^1}\,[\psi]_{\mathcal{C}^1_{[S,T]}L^\infty}\,h^{1-\varepsilon}\,(t-s)^{1+\frac{\varepsilon}{2}}$;

(ii) $\|\delta A_{s,u,t}\|_{L^m}\le C\,\|f\|_\infty\,h^{\frac12-\varepsilon}\,(t-s)^{\frac12+\frac{\varepsilon}{2}}$;

(iii) If (i) and (ii) are satisfied, (A.4) gives the convergence in probability of $\sum_{i=1}^{N_k-1}A_{t_i^k,t_{i+1}^k}$ along any sequence of partitions $\Pi_k=\{t_i^k\}_{i=1}^{N_k}$ of $[S,t]$ with mesh converging to 0. We will prove that the limit is the process $\mathcal{A}$ given in (5.3).

Assume for now that (i), (ii) and (iii) hold. Applying Lemma A.1, we obtain that

$$\begin{aligned}\Big\|\int_s^t f(\psi_r+B_r)-f(\psi_r+B_{r_h})\,dr\Big\|_{L^m}&\le C\Big(\|f\|_\infty h^{\frac12-\varepsilon}(t-s)^{\frac12+\frac{\varepsilon}{2}}+\|f\|_{\mathcal{C}^1}[\psi]_{\mathcal{C}^1_{[S,T]}L^\infty}h^{1-\varepsilon}(t-s)^{1+\frac{\varepsilon}{2}}\Big)\\ &\quad+\|A_{s,t}\|_{L^m}.\end{aligned}$$

We will see in (5.27) that $\|A_{s,t}\|_{L^m}\le C\,\|f\|_\infty h^{\frac12-\varepsilon}\,(t-s)^{\frac12+\frac{\varepsilon}{2}}$. Then choosing $(s,t)=(S,T)$, we get (5.23), using that $[\psi]_{\mathcal{C}^1_{[S,T]}L^\infty}\le[\psi]_{\mathcal{C}^1_{[0,1]}L^\infty}$.

We now check that the conditions (i), (ii) and (iii) actually hold.

Proof of (i): We have

$$\mathbb{E}^s\delta A_{s,u,t}=\mathbb{E}^s\int_u^t f(\psi_s+B_r)-f(\psi_s+B_{r_h})-f(\psi_u+B_r)+f(\psi_u+B_{r_h})\,dr.$$

**The case $t-u \leq 2h$.** In this case, using the Lipschitz norm of $f$, we have

$$|\mathbb{E}^s \delta A_{s,u,t}| \leq 2\|f\|_{\mathcal{C}^1} \int_u^t |\psi_s - \psi_u| \, dr.$$

Therefore using the inequality $(t-u)(u-s) \leq C\, h^{1-\varepsilon}\, (t-u)^{\varepsilon}(u-s) \leq C\,(t-s)^{1+\varepsilon} h^{1-\varepsilon}$,

$$\begin{aligned}\|\mathbb{E}^s \delta A_{s,u,t}\|_{L^m} &\leq \|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]} L^m} (t-u)(u-s) \\ &\leq C \|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]} L^\infty} (t-s)^{1+\varepsilon} h^{1-\varepsilon}.\end{aligned}$$

**The case $t-u > 2h$.** We split the integral between $u$ and $u+2h$ and then between $u+2h$ and $t$ as follows:

$$\begin{aligned}\mathbb{E}^s \delta A_{s,u,t} &= \int_u^{u+2h} \mathbb{E}^s [f(\psi_s + B_r) - f(\psi_s + B_{r_h}) - f(\psi_u + B_r) + f(\psi_u + B_{r_h})] \, dr \\ &\quad + \mathbb{E}^s \int_{u+2h}^t \mathbb{E}^u [f(\psi_s + B_r) - f(\psi_s + B_{r_h}) - f(\psi_u + B_r) + f(\psi_u + B_{r_h})] \, dr \\ &=: J_1 + J_2,\end{aligned}$$

using the tower property of conditional expectation for $J_2$. For $J_1$, we obtain from the case $t-u \leq 2h$ that

$$\|J_1\|_{L^m} = \|\mathbb{E}^s \delta A_{s,u,u+2h}\|_{L^m} \leq C \|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]} L^\infty} (t-s)^{1+\varepsilon} h^{1-\varepsilon}. \tag{5.24}$$

As for $J_2$, we use Lemma 3.3($i$) and (3.1) to write

$$\begin{aligned}J_2 &= \mathbb{E}^s \int_{u+2h}^t (G_{C_1(r-u)^{2H}} - G_{C_1(r_h-u)^{2H}})(f(\psi_s + \mathbb{E}^u B_r) - f(\psi_u + \mathbb{E}^u B_r)) \, dr \\ &\quad + \mathbb{E}^s \int_{u+2h}^t G_{C_1(r_h-u)^{2H}} \Big( f(\psi_s + \mathbb{E}^u B_r) - f(\psi_s + \mathbb{E}^u B_{r_h}) - f(\psi_u + \mathbb{E}^u B_r) + f(\psi_u + \mathbb{E}^u B_{r_h}) \Big) \, dr \\ &=: J_{21} + J_{22}.\end{aligned}$$

For $J_{21}$, we apply [6, Proposition 3.7 (ii)] with $\beta = 0$, $\delta = 1$, $\alpha = 0$ and $f \equiv f(\psi_s + \cdot) - f(\psi_u + \cdot)$ to get

$$\|J_{21}\|_{L^m} \leq C \|\mathbb{E}^s \|f(\psi_s + \cdot) - f(\psi_u + \cdot)\|_\infty \|_{L^m} \int_{u+2h}^t ((r-u)^{2H} - (r_h - u)^{2H})(r_h - u)^{-2H} \, dr.$$

Now using that $\|\mathbb{E}^s \|f(\psi_s+\cdot) - f(\psi_u+\cdot)\|_\infty\|_{L^m} \leq \|f\|_{\mathcal{C}^1} \|\mathbb{E}^s |\psi_s - \psi_u|\|_{L^m} \leq \|f\|_{\mathcal{C}^1} \|\psi_s - \psi_u\|_{L^\infty}$ and applying the inequalities $(r-u)^{2H} - (r_h-u)^{2H} \leq C(r-r_h)(r_h-u)^{2H-1}$ and $2(r_h-u) \geq (r-u)$, it comes

$$\begin{aligned}\|J_{21}\|_{L^m} &\leq C\, \|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]} L^\infty} |u-s| \int_{u+2h}^t (r-r_h)(r-u)^{2H-1}(r-u)^{-2H} \, dr \\ &\leq C\, \|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]} L^\infty}\, h\, (t-s)\, (|\log(2h)| + |\log(t-u)|).\end{aligned}$$

Since $t-u = (t-s)/2 > 2h$, one has

$$\|J_{21}\|_{L^m} \leq C\, \|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]} L^\infty}\, h^{1-\varepsilon} (t-s)^{1+\frac{\varepsilon}{2}}. \tag{5.25}$$

As for $J_{22}$, observe that

$$\begin{aligned}&\Big| G_{C_1(r_h-u)^{2H}} \Big( f(\psi_s + \mathbb{E}^u B_r) - f(\psi_s + \mathbb{E}^u B_{r_h}) - f(\psi_u + \mathbb{E}^u B_r) + f(\psi_u + \mathbb{E}^u B_{r_h}) \Big) \Big| \\ &\quad \leq \|G_{C_1(r_h-u)^{2H}} f(\psi_s + \cdot) - G_{C_1(r_h-u)^{2H}} f(\psi_u + \cdot)\|_{\mathcal{C}^1}\, |\mathbb{E}^u B_r - \mathbb{E}^u B_{r_h}| \\ &\quad \leq C\, \|f(\psi_s + \cdot) - f(\psi_u + \cdot)\|_\infty\, (r_h - u)^{-H}\, |\mathbb{E}^u B_r - \mathbb{E}^u B_{r_h}| \\ &\quad \leq C\, \|f\|_{\mathcal{C}^1}\, (r_h - u)^{-H}\, |\psi_s - \psi_u|\, |\mathbb{E}^u B_r - \mathbb{E}^u B_{r_h}|,\end{aligned}$$

where we used [6, Proposition 3.7 (i)] with $\beta = 1$ and $\alpha = 0$ in the penultimate inequality. Now in view of the previous inequality, using consecutively Jensen's inequality, [6, Proposition 3.6 (v)], that $2(r_h - u) \geq r - u$ and that $t - u = (t-s)/2 > 2h$, it comes

$$\begin{aligned}
\|J_{22}\|_{L^m} &\leq C\,\|f\|_{\mathcal{C}^1} \int_{u+2h}^t \|\mathbb{E}^s[|\psi_s - \psi_u|\,|\mathbb{E}^u(B_r - B_{r_h})|]\|_{L^m} (r_h - u)^{-H}\,dr \\
&\leq C\,\|f\|_{\mathcal{C}^1}\,\|\psi_s - \psi_u\|_{L^\infty} \int_{u+2h}^t \|\mathbb{E}^u(B_r - B_{r_h})\|_{L^m} (r_h - u)^{-H}\,dr \\
&\leq C\,\|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]}L^\infty}\,(u-s) \int_{u+2h}^t (r - r_h)(r-u)^{H-1}(r_h - u)^{-H}\,dr \\
&\leq C\,\|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]}L^\infty}\,h\,(t-s)\,(|\log(2h)| + |\log(t-u)|) \\
&\leq C\,\|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]}L^\infty}\,h^{1-\varepsilon}\,(t-s)^{1+\frac{\varepsilon}{2}}. \qquad (5.26)
\end{aligned}$$

In view of the inequalities (5.24), (5.25) and (5.26), we have finally

$$\|\mathbb{E}^s \delta A_{s,u,t}\|_{L^m} \leq C\,\|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[S,T]}L^\infty} h^{1-\varepsilon} (t-s)^{1+\frac{\varepsilon}{2}}.$$

Proof of (ii): We write

$$\|\delta A_{s,u,t}\|_{L^m} \leq \|A_{s,t}\|_{L^m} + \|A_{s,u}\|_{L^m} + \|A_{u,t}\|_{L^m}$$

and we apply Lemma 5.10 for each term in the right-hand side of the previous inequality, respectively for $\psi = \psi_s$, $\psi_s$ again and $\psi_u$. We thus have

$$\|A_{s,t}\|_{L^m} \leq C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}, \qquad (5.27)$$

and combining similar inequalities on $A_{s,u}$ and $A_{u,t}$ with (5.27) yields

$$\|\delta A_{s,u,t}\|_{L^m} \leq C\,\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

Proof of (iii): Finally, for a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[S,t]$ with $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$ and mesh size $|\Pi_k|$ converging to zero, we have

$$\begin{aligned}
\Big\|\mathcal{A}_t - \sum_{i=1}^{N_k-1} A_{t_i,t_{i+1}}\Big\|_{L^1} &\leq \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \mathbb{E}|f(\psi_r + B_r) - f(\psi_r + B_{r_h}) - f(\psi_{t_i^k} + B_r) + f(\psi_{t_i^k} + B_{r_h})|\,dr \\
&\leq C \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} \|f\|_{\mathcal{C}^1} \|\psi_r - \psi_{t_i^k}\|_{L^1}\,dr \\
&\leq C\|f\|_{\mathcal{C}^1} \sum_{i=1}^{N_k-1} \int_{t_i^k}^{t_{i+1}^k} [\psi]_{\mathcal{C}^1_{[0,1]}L^\infty} |\Pi_k|\,dr \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

□

**Corollary 5.12.** *Let $\varepsilon \in (0, \frac{1}{2})$ and $m \in [2,\infty)$. There exists a constant $C > 0$ such that for any $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process $(\psi_t)_{t\in[0,1]}$, any $f \in \mathcal{C}_b^1(\mathbb{R}^d, \mathbb{R}^d)$, any $h \in (0,1)$, and any $(s,t) \in \Delta_{0,1}$,*

$$\Big\|\int_s^t f(\psi_r + B_r) - f(\psi_{r_h} + B_{r_h})\,dr\Big\|_{L^m} \leq C\left(\|f\|_\infty\, h^{\frac{1}{2}-\varepsilon}(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}} + \|f\|_{\mathcal{C}^1}[\psi]_{\mathcal{C}^1_{[0,1]}L^\infty}\, h^{1-\varepsilon}(t-s)\right).$$

*Proof.* Introducing the pivot term $f(\psi_r + B_{r_h})$, we have

$$\begin{aligned}
&\Big\|\int_s^t f(\psi_r + B_r) - f(\psi_{r_h} + B_{r_h})\,dr\Big\|_{L^m} \\
&\leq \Big\|\int_s^t f(\psi_r + B_{r_h}) - f(\psi_{r_h} + B_{r_h})\,dr\Big\|_{L^m} + \Big\|\int_s^t f(\psi_r + B_r) - f(\psi_r + B_{r_h})\,dr\Big\|_{L^m} \\
&=: J_1 + J_2.
\end{aligned}$$

We first bound $J_1$ using the $\mathcal{C}^1$ norm of $f$:

$$J_1 \leq \|f\|_{\mathcal{C}^1} \int_s^t \|\psi_r - \psi_{r_h}\|_{L^m} \, dr \leq \|f\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[0,1]} L^\infty} \, h \, (t-s).$$

Then $J_2$ is bounded by Proposition 5.11. Combining the two bounds, we get the desired result. □

**Corollary 5.13.** *Recall that the process $K^{h,n}$ was defined in (4.1). Let $\varepsilon \in (0, \frac{1}{2})$ and $m \in [2, \infty)$. There exists a constant $C > 0$ such that for any $(s,t) \in \Delta_{0,1}$, any $h \in (0,1)$ and any $n \in \mathbb{N}$, we have*

$$\Big\| \int_s^t b^n(X_0 + K_r^{h,n} + B_r) - b^n(X_0 + K_{r_h}^{h,n} + B_{r_h}) \, dr \Big\|_{L^m} \leq C \Big( \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1} \|b^n\|_\infty h^{1-\varepsilon} \Big) (t-s)^{\frac{1}{2}}.$$

*Proof.* Define the process $\psi_t = X_0 + K_t^{h,n}$, $t \in [0,1]$. Since $\psi$ is $\mathbb{F}$-adapted and $b^n \in \mathcal{C}_b^1(\mathbb{R}^d, \mathbb{R}^d)$, we apply Corollary 5.12 to get

$$\begin{aligned} &\Big\| \int_s^t b^n(X_0 + K_r^{h,n} + B_r) - b^n(X_0 + K_{r_h}^{h,n} + B_{r_h}) \, dr \Big\|_{L^m} \\ &\leq C \Big( \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} (t-s)^{\frac{\varepsilon}{2}} + \|b^n\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[0,1]} L^\infty} h^{1-\varepsilon} (t-s)^{\frac{1}{2}} \Big) (t-s)^{\frac{1}{2}} \\ &\leq C \Big( \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1} [\psi]_{\mathcal{C}^1_{[0,1]} L^\infty} h^{1-\varepsilon} \Big) (t-s)^{\frac{1}{2}}. \end{aligned}$$

It remains to prove an upper bound on $[\psi]_{\mathcal{C}^1_{[0,1]} L^\infty}$. For $0 \leq u \leq v \leq 1$, we have

$$|\psi_v - \psi_u| = |\int_u^v b^n(X_{r_h}^{h,n}) \, dr| \leq |v-u| \|b^n\|_\infty.$$

Hence $[\psi]_{\mathcal{C}^1_{[0,1]} L^\infty} \leq \|b^n\|_\infty$. □

# 6 Examples and simulations

In this section, we discuss examples of SDEs of the form (1.1) that can be treated by Theorem 2.5.

## 6.1 Skew fractional Brownian motion

The skew Brownian motion is a one-dimensional process that behaves like a Brownian motion with a certain diffusion coefficient above the $x$-axis, and with another diffusion coefficient below the $x$-axis. We refer to [19, 27] for various constructions, and in particular in [26], it is shown to be the solution of an SDE which involves its local time. This equation reads $dX_t = \alpha \, dL_t^X + dW_t$, for $\alpha \in (-1,1)$, where $L^X$ is the local time at 0 of the solution. Formally we can write $dL_t^X = \delta_0(X_t) \, dt$. More generally in $\mathbb{R}^d$, although this is not the only possible approach (see e.g. [4, 16] for alternative definitions), we call skew fractional Brownian motion the solution to (1.1) when the drift is $\alpha\delta_0$, $\alpha \in \mathbb{R}^d$, that is

$$dX_t = \alpha \delta_0(X_t) \, dt + dB_t. \tag{6.1}$$

Since $\delta_0 \in \mathcal{B}_p^{-d+\frac{d}{p}}$, Theorem 2.3 gives strong existence and uniqueness for $H < \frac{1}{2(d+1)}$ and the tamed Euler scheme converges for the same values of $H$ by Theorem 2.5.

**Remark 6.1.** *Since $H < 1/(2(d+1)) < 1/d$, we know from [42, Theorem 7.1] that the fBm visits any state $x$ (and in particular 0) infinitely may times with positive probability. So the equation (6.1) is not simply reduced to $X = B$.*
*Instead of putting a Dirac measure in dimension $d > 1$, one can also define a skew fBm on some set $S$ of dimension $d-1$ by considering a measure $\mu$ supported on $S$ (for example a Hausdorff measure). As before, we know from [42, Theorem 7.1] that for $H < 1/d$, the fBm visits $S$ infinitely*

*many times with positive probability. So the equation $dX_t = \mu(X_t)dt + dB_t$ is not reduced to $X = B$. Since signed measures also belong to $\mathcal{B}_p^{-d+d/p}$ [5, Proposition 2.39], we have again strong existence and uniqueness for $H < 1/(2(d+1))$.*

As an alternative construction of the skew fBm, we also propose to replace the local time by its approximation $b(x) = \frac{\alpha}{2\varepsilon}\mathbb{1}_{(-\varepsilon,\varepsilon)}(x)$, $\varepsilon > 0$. Now we have $b$ bounded and so $b \in \mathcal{B}_\infty^0$, therefore one can take $H < 1/2$ and consider the SDE

$$dX_t = \frac{\alpha}{(2\varepsilon)^d}\mathbb{1}_{(-\varepsilon,\varepsilon)^d}(X)\,dt + dB_t.$$

In the Markovian case and dimension $d = 1$, the skew Brownian motion is reflected on the $x$-axis when $\alpha = \pm 1$. Unlike the skew Brownian motion, the skew fBm (for $H \neq 1/2$) is not reflected for any value of $\alpha$, since $X - B$ is more regular than $B$ (see Theorem 2.3). To construct reflected processes, a classical approach is to proceed by penalization, see e.g. [28] in the Brownian case, and [35] for rough differential equations. This consists in choosing a drift of the form $b_\varepsilon(x) = \frac{(x)_-}{\varepsilon}$ and letting $\varepsilon$ tend to 0. Note that this approach also works for stochastic partial differential equations (SPDEs), see for instance [18, 32, 43]. If we consider more specifically the stochastic heat equation, the solution in time observed at a fixed point in space behaves qualitatively like a fractional SDE with Hurst parameter $H = \frac{1}{4}$. Hence it is interesting to consider the following one-dimensional SDE:

$$dX_t^\varepsilon = \frac{(X_t^\varepsilon)_-}{\varepsilon}\kappa(X_t^\varepsilon)\,dt + dB_t. \tag{6.2}$$

In [35], $\kappa$ was essentially the identity mapping and the distance between $X^\varepsilon$ and $X^0$ was quantified, with $X^0$ a reflected process. But then the drift is not in some Besov space. So in order to approximate (6.2) numerically, we could assume that $\kappa$ is a smooth cut-off to ensure that the drift is in some Besov space (e.g. $\mathcal{B}_\infty^1$), however it is no longer clear that $X^\varepsilon$ converges to a reflected process. We leave the question of numerical approximation of reflected fractional processes for future research.

## 6.2 Applications in finance

Some models of mathematical finance involve irregular drifts.

First, consider a dividend paying firm, whose capital evolution can be modelled by the following one-dimensional SDE:

$$dX_t = (r - \mathbb{1}_{X_t \leq q})\,dt + \sigma\,dB_t,$$

with an interest rate $r$, the volatility of the market $\sigma$ and some threshold $q$, see e.g [2].

**Remark 6.2.** *An extension of the previous SDE to dimension $d$ can be done by considering a threshold of the form $\mathbb{1}_{x\in D}$ where $D$ is some domain in $\mathbb{R}^d$ or $\mathbb{1}_{f(x)\leq q}$ where $f : \mathbb{R}^d \mapsto \mathbb{R}$.*

Numerical methods for bounded drifts with Brownian noise exist in the literature, see e.g. [8, 21]. When $B$ is a fractional Brownian motion with $H < 1/2$, [6] provides a rate of convergence for the strong error (and Theorem 2.5 provides the same rate of convergence).

Then, we propose a class of models which can be related heuristically to the rough Heston model introduced in [11]. Recently, it was observed empirically that the volatility in some high-frequency financial markets has a very rough behaviour, in the sense that its trajectories have a very small Hölder exponent, close to 0.1. Formally, the volatility component in the rough Heston model is described by a square root diffusion coefficient and a very rough driving noise. It would read

$$dV_t = \kappa(V_t)\,dt + \sqrt{V_t}\,dB_t, \tag{6.3}$$

if we could make sense of this equation, the difficulty being both to define a stochastic integral when $H$ is small, and to ensure the positivity of the solution. Note that it is possible to define properly a rough Heston model, by means of Volterra equations, see [11]. However, we keep discussing (6.3) at a formal level, and consider the Lamperti transform $L(x) = \sqrt{x}$. Assume that a first order chain rule holds for the solution of (6.3), then as long as $V$ stays nonnegative, it comes that

$$L(V_t) = L(V_0) + \int_0^t \frac{\kappa(V_s)}{2\sqrt{V_s}}\, ds + \frac{1}{2} B_t,$$

which for $\widetilde{V}_t := L(V_t) = \sqrt{V_t}$ also reads

$$\widetilde{V}_t = \widetilde{V}_0 + \frac{1}{2}\int_0^t \frac{1}{\widetilde{V}_s}\kappa(\widetilde{V}_s^2)\, ds + \frac{1}{2} B_t. \tag{6.4}$$

While there are some quantitative numerical approximation results for rough models (e.g. for the rough Bergomi model [13, 17]), the Euler scheme for the rough Heston model is only known to converge without a rate [36]. Now we can make sense of the Equation (6.4) with drift $b(x) = \frac{\kappa(x^2)}{2|x|^{1-\varepsilon}}$ as for $\kappa$ a bump function and for small $\varepsilon > 0$, $b \in \mathcal{B}_1^0$ (see [5, Prop. 2.21]). Hence Theorem 2.5 can be applied whenever $H < 1/4$ and in view of Corollary 2.6, this yields a strong error of order $(1/4)^-$.

## 6.3 Fractional Bessel processes (in dimension 1)

Bessel processes [34, Chapter XI] play an important role in probability theory and financial mathematics.

As a generalization and motivated by the discussion in the previous subsection, we consider solutions to the following one-dimensional SDE:

$$dX_t = \frac{\kappa(X_t)}{|X_t|^\alpha}\, dt + dB_t, \tag{6.5}$$

for some $\alpha > 0$ and $H \in (0, 1)$. When $H = 1/2$, $\alpha = 1$ and $\kappa$ is the identity function, we know that the solution always stays positive [34, Chapter XI, Section 1]. By computations similar to [5, Prop. 2.21], the drift $b(x) = \kappa(x)\,|x|^{-\alpha}$ belongs to $\mathcal{B}_\infty^{-\alpha}$ for $\alpha \in (0, 1)$. In this case, (H2) reads $H < \frac{1}{2(1+\alpha)}$ and the rate of convergence of the tamed Euler scheme is close to $\frac{1}{2(1+\alpha)}$.

## 6.4 Other examples in higher dimension

A way to extend processes (6.5) to dimension 2 could be the following:

$$dX_t^i = \frac{\kappa(X_t)}{|X_t|^\alpha}\, dt + dB_t^i,\ i = 1, 2, \tag{6.6}$$

where $B^1$ and $B^2$ are two independent fBms and $\alpha > 0$. By [5, Proposition 2.21], one can prove that $x \mapsto b(x) = \frac{\kappa(x)}{|x|^\alpha}$ belongs to $\mathcal{B}_\infty^{-\alpha}$ for $\alpha \in (0, 2)$. Therefore, the condition on $H$ becomes $H < \frac{1}{2(1+\alpha)}$.
Notice that the SDE (6.6) presents a singularity only at the point $(0, 0)$. To create a singularity on both the $x$ and $y$-axes, one could also look at the following SDE

$$dX_t^i = \frac{1}{(|X_t^1| \wedge |X_t^2|)^\alpha}\, dt + dB_t^i,\ i = 1, 2.$$

Another example to consider in higher dimension is an SDE with discontinuous drift. For instance, let the drift be an indicator function of some domain $D$ as in (6.7):

$$dX_t = \mathbb{1}_D^{(d)}(X_t)\, dt + dB_t, \tag{6.7}$$

where $\mathbb{1}_D^{(d)}$ denotes the vector-valued indicator function with identical entries $\mathbb{1}_D$ on each component. We have $\mathbb{1}_D^{(d)} \in \mathcal{B}_\infty^0$, and thus one can take $H < 1/2$.

### 6.5 Simulations

In dimension 1, we will simulate two SDEs. First the skew fractional Brownian motion (6.1) with $\alpha = 1$. Then we simulate the SDE with bounded measurable drift $\mathbb{1}_{\mathbb{R}_+} \in \mathcal{B}^0_\infty$, i.e.

$$dX_t = \mathbb{1}_{X_t>0}\, dt + dB_t. \tag{6.8}$$

The drifts are approximated by convolution with the Gaussian kernel, that is $b^n(x) = G_{\frac{1}{n}} b(x)$ and we fix the initial condition to $X_0 = 0$. For the skew fBm, this corresponds to

$$b^n(x) = \sqrt{\frac{n}{2\pi}} e^{-\frac{nx^2}{2}},$$

and for (6.8) this yields

$$b^n(x) = \sqrt{\frac{n}{2\pi}} \int_0^x e^{-\frac{ny^2}{2}}\, dy.$$

As in Corollary 2.6, we fix the parameter $n$ of the mollifier in the tamed Euler scheme as $n = \lfloor h^{-\frac{1}{1-\gamma+\frac{d}{p}}} \rfloor$. Our aim is to observe the rate of convergence numerically, so we need a reference value for the solutions of (6.1) and (6.8). However these solutions do not have an explicit expression so we do not have an exact reference value. Instead, we first make a costly computation with very small time-step $h = 2^{-7}\, 10^{-4}$ that will serve as reference value. In a second step, we compute the tamed Euler scheme for $h \in \{2^{-1}\, 10^{-4},\ 2^{-2}\, 10^{-4},\ 2^{-3}\, 10^{-4},\ 2^{-4}\, 10^{-4}\}$ and compare it to the reference value with the same noise and $h = 2^{-7}\, 10^{-4}$. The result is averaged over $N = 50000$ realisations of the noise to get an estimate of the strong error.

In dimension 2, we simulate the 2-dimensional SDE (6.7) with $X_0 = 0$ and with $D$ the quadrant defined by $D = \{x = (x_1, x_2) :\ x_1 \geq 0, x_2 \geq 0\}$. The drift $b$ is approximated by

$$b^n(x) = G_{\frac{1}{n}} b(x) = \frac{n}{2\pi} \int_{\mathbb{R}^2} e^{-\frac{n}{2}|x-y|^2} \mathbb{1}_D(y)\, dy.$$

Recall that according to Corollary 2.6, the theoretical order of convergence is almost 1/2 when the drift is bounded and almost 1/4 when the drift is a Dirac distribution. We plot the logarithmic strong error with respect to the time-step $h$ for several values of the Hurst parameter, in Figure 1 for the Equations (6.8) and (6.7), and in Figure 2 for the Equation (6.1). We conclude that the empirical order of convergence is consistent with the theoretical one.

# Appendices

## A Proofs of regularisation by fBm in dimension $d$

We start by recalling an extension of the stochastic sewing Lemma [24] with singular weights that was established in [3]. It is useful for the main estimates of Section 3 (Lemma 3.4 and Proposition 3.5, whose proofs are developed in this appendix) and also in Section 5.

For $\alpha \in [0, 1)$ and $(s, t) \in \Delta_{S,T}$ we define

$$\nu^{(\alpha)}_{S,T}(s, t) := \int_s^t (r - S)^{-\alpha} dr,$$

which satisfies

$$\nu^{(\alpha)}_{S,T}(s, t) \leq C\, (t - s)^{1-\alpha}. \tag{A.1}$$

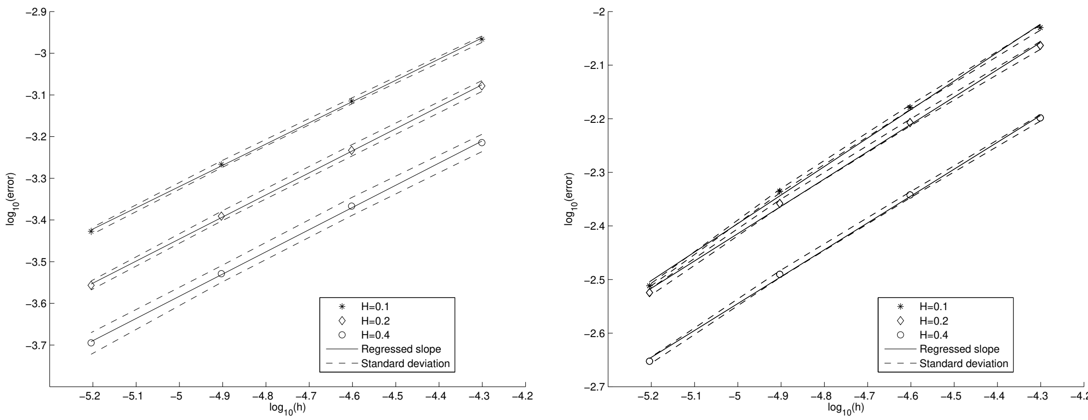


Figure 1: Plot of the logarithm of the strong error ($y$-axis) against $h$ ($x$-axis) for a bounded drift. Left: Equation (6.8) ($d = 1$) - Right: Equation (6.7) ($d = 2$) . For different values of $H < 1/2$, and in both dimension 1 and 2, we observe that the numerical order of convergence (by linear regression) is approximately 0.5 (with a standard deviation plotted in dashed lines), which coincides with the theoretical order 1/2.

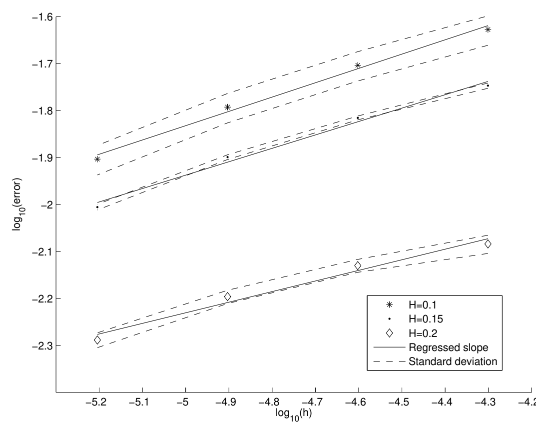


Figure 2: Plot of the logarithm of the strong error ($y$-axis) against $h$ ($x$-axis) for a Dirac drift in dimension $d = 1$, namely Equation (6.1). For several values of $H < 1/2$, we observe that the numerical order of convergence (by linear regression) is approximately 0.25 (with a standard deviation plotted in dashed lines), which is close to the theoretical order 1/4.

**Lemma A.1** ([3])**.** *Let $0 \leq S < T$, $m \in [2, \infty)$ and $q \in [m, \infty]$. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Let $A : \Delta_{S,T} \to L^m$ be such that $A_{s,t}$ is $\mathcal{F}_t$-measurable for any $(s,t) \in \Delta_{S,T}$. Assume that there exist constants $\Gamma_1, \Gamma_2 \geq 0$, $\alpha_1 \in [0,1)$, $\alpha_2 \in [0, \frac{1}{2})$ and $\varepsilon_1, \varepsilon_2 > 0$ such that for any $(s,t) \in \Delta_{S,T}$ and $u := (s+t)/2$,*

$$\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^q} \leq \Gamma_1 \, (u-S)^{-\alpha_1} (t-s)^{1+\varepsilon_1}, \tag{A.2}$$

$$\|(\mathbb{E}^S|\delta A_{s,u,t}|^m)^{\frac{1}{m}}\|_{L^q} \leq \Gamma_2 \, (u-S)^{-\alpha_2} (t-s)^{\frac{1}{2}+\varepsilon_2}. \tag{A.3}$$

*Then there exists a process $(\mathcal{A}_t)_{t \in [S,T]}$ such that, for any $t \in [S,T]$ and any sequence of partitions $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ of $[S,t]$ with mesh size going to zero, we have*

$$\mathcal{A}_t = \lim_{k \to \infty} \sum_{i=0}^{N_k} A_{t_i^k, t_{i+1}^k} \text{ in probability.} \tag{A.4}$$

*Moreover, there exists a constant $C = C(\varepsilon_1, \varepsilon_2, m, \alpha_1, \alpha_2)$ independent of $S, T$ such that for* *every* $(s,t)\in\Delta_{S,T}$ *we have*

$$\|(\mathbb{E}^S|\mathcal{A}_t-\mathcal{A}_s-A_{s,t}|^m)^{\frac{1}{m}}\|_{L^q}\leq C\,\Gamma_1\nu_{S,T}^{(\alpha_1)}(s,t)(t-s)^{\varepsilon_1}+C\,\Gamma_2\Big(\nu_{S,T}^{(2\alpha_2)}(s,t)\Big)^{\frac{1}{2}}(t-s)^{\varepsilon_2},$$

*and*

$$\|\mathbb{E}^S[\mathcal{A}_t-\mathcal{A}_s-A_{s,t}]\|_{L^q}\leq C\,\Gamma_1\,\nu_{S,T}^{(\alpha_1)}(s,t)\,(t-s)^{\varepsilon_1}.$$

**Remark A.2.**

- *In this paper, the stochastic sewing Lemma is applied for only two possible values of* $q$*, that is* $q=\infty$ *or* $q=m$*, in which case we have* $\|(\mathbb{E}^S|\cdot|^m)^{\frac{1}{m}}\|_{L^m}=\|\cdot\|_{L^m}$*.*
- *A critical-exponent version of the stochastic sewing Lemma, introduced in [3, Theorem 4.5] is used in Proposition 5.4. Under the same notations and assumptions as Lemma A.1 (with* $q=m$ *and* $\alpha_1=\alpha_2=0$*), assuming moreover that there exist* $\Gamma_3,\Gamma_4,\varepsilon_4>0$ *such that*

$$\|\mathbb{E}^s\left[\delta A_{s,u,t}\right]\|_{L^m}\leq\Gamma_3|t-s|+\Gamma_4|t-s|^{1+\varepsilon_4},\tag{A.5}$$

*we get that for* $(s,t)\in\Delta_{S,T}$*,*

$$\|\mathcal{A}_t-\mathcal{A}_s-A_{s,t}\|_{L^m}\leq C\Gamma_3\left(1+\left|\log\frac{\Gamma_1T^{\varepsilon_1}}{\Gamma_3}\right|\right)(t-s)+C\Gamma_2(t-s)^{\frac{1}{2}+\varepsilon_2}+C\Gamma_4(t-s)^{1+\varepsilon_4}.$$

## A.1 Proof of Lemma 3.4

We will apply Lemma A.1 for $S\leq s\leq t\leq T$,

$$\mathcal{A}_t:=\int_S^t f(B_r,\Xi)\,dr\quad\text{and}\quad A_{s,t}:=\mathbb{E}^s\left[\int_s^t f(B_r,\Xi)\,dr\right].$$

Notice that we have $\mathbb{E}^s[\delta A_{s,u,t}]=0$, so (A.2) trivially holds. In order to establish (A.3), we will show that for some $\varepsilon_2>0$,

$$\|\delta A_{s,u,t}\|_{L^q}\leq\Gamma_2\,(t-s)^{\frac{1}{2}+\varepsilon_2}(u-S)^{-\frac{dH}{p}}.\tag{A.6}$$

For $u=(s+t)/2$ we have by the triangle inequality, Jensen's inequality for conditional expectation and Lemma 3.3($iv$) (recall that $q\leq p$) that

$$\begin{aligned}\|\delta A_{s,u,t}\|_{L^q}&\leq\left\|\mathbb{E}^s\left[\int_u^t f(B_r,\Xi)\,dr\right]\right\|_{L^q}+\left\|\mathbb{E}^u\left[\int_u^t f(B_r,\Xi)\,dr\right]\right\|_{L^q}\\&\leq\int_u^t(\|\mathbb{E}^sf(B_r,\Xi)\|_{L^q}+\|\mathbb{E}^uf(B_r,\Xi)\|_{L^q})\,dr\\&\leq 2\int_u^t\|\mathbb{E}^uf(B_r,\Xi)\|_{L^q}\,dr\\&\leq C\int_u^t\|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\beta}\|_{L^q}(r-u)^{H\beta}(u-S)^{-\frac{d}{2p}}(r-S)^{d\frac{1-2H}{2p}}\,dr\\&\leq C\,\|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\beta}\|_{L^q}\,(t-u)^{1+H\beta}(u-S)^{-\frac{dH}{p}},\end{aligned}$$

where we used $r-S\leq 2(u-S)$ for the last inequality. Hence, we have (A.6) for $\varepsilon_2=1/2+H\beta>0$.

Let $t\in[S,T]$. Let $(\Pi_k)_{k\in\mathbb{N}}$ be a sequence of partitions of $[S,t]$ with mesh size converging to zero. For each $k$, denote $\Pi_k=\{t_i^k\}_{i=1}^{N_k}$. By Lemma 3.3($iii$) we have that

$$\begin{aligned}\left\|\mathcal{A}_t-\sum_i A_{t_i^k,t_{i+1}^k}\right\|_{L^1}&\leq\sum_i\int_{t_i^k}^{t_{i+1}^k}\|f(B_r,\Xi)-\mathbb{E}^{t_i^k}f(B_r,\Xi)\|_{L^1}dr\\&\leq C\,\|\|f(\cdot,\Xi)\|_{\mathcal{C}^1}\|_{L^2}\,(t-S)\,|\Pi_k|^H\longrightarrow 0.\end{aligned}$$

Hence (A.4) holds true.

Applying Lemma A.1, we get

$$\|(\mathbb{E}^S|\mathcal{A}_t-\mathcal{A}_s|^m)^{\frac{1}{m}}\|_{L^q}\leq\|A_{s,t}\|_{L^q}+C\,\|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\beta}\|_{L^q}\left(\nu_{S,T}^{(\frac{2dH}{p})}(s,t)\right)^{\frac{1}{2}}(t-s)^{\frac{1}{2}+H\beta}. \tag{A.7}$$

To bound $\|A_{s,t}\|_{L^q}$, notice that

$$\|A_{s,t}\|_{L^q}=\left\|\mathbb{E}^s\int_s^t f(B_r,\Xi)\,dr\right\|_{L^q}\leq\int_s^t\|\mathbb{E}^s f(B_r,\Xi)\|_{L^q}dr. \tag{A.8}$$

Hence to obtain (3.2), use Lemma 3.3($ii$) and recall that $1+H(\beta-\frac{d}{p})>0$ to get that

$$\begin{aligned}\|A_{s,t}\|_{L^q}&\leq C\int_s^t\|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\beta}\|_{L^q}\,(r-s)^{H(\beta-\frac{d}{p})}dr\\&\leq C\,\|\|f(\cdot,\Xi)\|_{\mathcal{B}_p^\beta}\|_{L^q}\,(t-s)^{1+H(\beta-\frac{d}{p})}.\end{aligned}$$

Plugging the previous inequality in (A.7) with the first inequality of (A.1) yields (3.2).

## A.2 Proof of Proposition 3.5

Let $(S,T)\in\Delta_{0,1}$. For $(s,t)\in\Delta_{S,T}$, let

$$A_{s,t}:=\int_s^t f(B_r+\psi_s)dr\text{ and }\mathcal{A}_t:=\int_S^t f(B_r+\psi_r)dr. \tag{A.9}$$

**Proof of (a).** Assume that $[\psi]_{\mathcal{C}^\tau_{[S,T]}L^{m,q}}<\infty$, otherwise (3.3) trivially holds. In the last part of this proof, we will check that the conditions in order to apply Lemma A.1 are verified. Namely, we will show that (A.2) and (A.3) hold true with $\varepsilon_1=H(\beta-d/p-1)+\tau>0$, $\alpha_1=0$ and $\varepsilon_2=1/2+H(\beta-\frac{d}{p})>0$, $\alpha_2=0$, so that there exists a constant $C>0$ independent of $S,T,s,t$ such that

(i$_a$) $\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^q}\leq C\,\|f\|_{\mathcal{B}_p^\beta}\,[\psi]_{\mathcal{C}^\tau_{[S,T]}L^{m,q}}\,(t-s)^{1+H(\beta-\frac{d}{p}-1)+\tau}$;

(ii$_a$) $\left\|(\mathbb{E}^S|\delta A_{s,u,t}|^m)^{\frac{1}{m}}\right\|_{L^q}\leq C\,\|f\|_{\mathcal{B}_p^\beta}(t-s)^{1+H(\beta-\frac{d}{p})}$;

(iii$_a$) If (i$_a$) and (ii$_a$) are satisfied, (A.4) gives the convergence in probability of $\sum_{i=0}^{N_k-1}A_{t_i^k,t_{i+1}^k}$ along any sequence of partitions $\Pi_k=\{t_i^k\}_{i=0}^{N_k}$ of $[S,t]$ with mesh converging to 0. We will prove that the limit is the process $\mathcal{A}$ given in (A.9).

Assume for now that (i$_a$), (ii$_a$) and (iii$_a$) hold. Applying Lemma A.1 and recalling (A.1), we obtain that

$$\begin{aligned}\left\|\left(\mathbb{E}^S\left|\int_s^t f(B_r+\psi_r)\,dr\right|^m\right)^{\frac{1}{m}}\right\|_{L^q}&\leq C\,\|f\|_{\mathcal{B}_p^\beta}\,[\psi]_{\mathcal{C}^\tau_{[S,T]}L^{m,q}}\,(t-s)^{1+H(\beta-1-\frac{d}{p})+\tau}\\&\quad+C\,\|f\|_{\mathcal{B}_p^\beta}(t-s)^{1+H(\beta-\frac{d}{p})}+\|(\mathbb{E}^S|A_{s,t}|^m)^{\frac{1}{m}}\|_{L^q}.\end{aligned}$$

To bound $\|(\mathbb{E}^S|A_{s,t}|^m)^{\frac{1}{m}}\|_{L^q}$, we apply Lemma 3.4 to $\Xi=\psi_s$. As $f$ is smooth and bounded, the first assumption of Lemma 3.4 is verified. By Lemma 3.1($i$), $\|f(\cdot+\psi_s)\|_{\mathcal{B}_p^\beta}\leq\|f\|_{\mathcal{B}_p^\beta}$, hence the second assumption of Lemma 3.4 is verified. It follows by Lemma 3.4 that

$$\begin{aligned}\|(\mathbb{E}^S|A_{s,t}|^m)^{\frac{1}{m}}\|_{L^q}&\leq C\,\|\|f(\psi_s+\cdot)\|_{\mathcal{B}_p^\beta}\|_{L^q}\,(t-s)^{1+H(\beta-\frac{d}{p})}\\&\leq C\,\|f\|_{\mathcal{B}_p^\beta}\,(t-s)^{1+H(\beta-\frac{d}{p})}.\end{aligned} \tag{A.10}$$

Then, we get (3.3).

**Proof of (b).** Assume that $[\psi]_{\mathcal{C}^{1/2+H}_{[S,T]}L^m} < \infty$, otherwise (3.4) trivially holds. In the last part of this proof, we will check that the conditions in order to apply the stochastic sewing Lemma with critical exponent [3, Theorem 4.5] are verified. Namely, we will show that for some $\varepsilon \in (0,1)$ small enough (specified later), (A.2), (A.3) and (A.5) hold true with $\varepsilon_1 = H > 0$, $\alpha_1 = 0$, $\varepsilon_2 = \varepsilon/2 > 0$, $\alpha_2 = 0$ and $\Gamma_4 = 0$, so that there exists a constant $C > 0$ independent of $s, t, S$ and $T$ such that

(i$_b$) $\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \le C\,\|f\|_{\mathcal{B}^{\beta+1}_p}\,[\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^m}\,(t-s)^{1+H}$;

(i$'_b$) $\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \le C\,\|f\|_{\mathcal{B}^{\beta}_p}\,[\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^m}\,(t-s)$ ;

(ii$_b$) $\|\delta A_{s,u,t}\|_{L^m} \le C\,\|f\|_{\mathcal{B}^{\beta}_p}\Big(1+[\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^m}\Big)(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}$;

(iii$_b$) If (i$_b$) and (ii$_b$) are satisfied, (A.4) gives the convergence in probability of $\sum_{i=0}^{N_k-1} A_{t_i^k,t_{i+1}^k}$ along any sequence of partitions $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ of $[S,t]$ with mesh converging to 0. We will prove that the limit is the process $\mathcal{A}$ given in (A.9).

Assume for now that (i$_b$), (i$'_b$), (ii$_b$) and (iii$_b$) hold. Applying [3, Theorem 4.5], we obtain that

$$\begin{aligned}\Big\|\int_s^t f(B_r+\psi_r)\,dr\Big\|_{L^m} &\le C\,\|f\|_{\mathcal{B}^{\beta}_p}\,[\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^m}\Big(1+\Big|\log\frac{\|f\|_{\mathcal{B}^{\beta+1}_p}t^{\varepsilon_1}}{\|f\|_{\mathcal{B}^{\beta}_p}}\Big|\Big)(t-s)\\ &\quad + C\,\|f\|_{\mathcal{B}^{\beta}_p}\Big(1+[\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^m}\Big)(t-s)^{\frac{1}{2}+\frac{\varepsilon}{2}}\\ &\quad + \|A_{s,t}\|_{L^m}.\end{aligned}$$

To bound $\|A_{s,t}\|_{L^m}$, we use again (A.10) with $\beta - \frac{d}{p} = -\frac{1}{2H}$ to get $\|A_{s,t}\|_{L^m} \le C\|f\|_{\mathcal{B}^{\beta}_p}(t-s)^{\frac{1}{2}}$. Hence we get (3.4).

We now check that the conditions (i$_a$), (ii$_a$), (iii$_a$), (i$_b$), (i$'_b$), (ii$_b$) and (iii$_b$) actually hold.

**Proof of (i$_a$), (i$_b$) and (i$'_b$).** For $(s,t) \in \Delta_{S,T}$, we have

$$\delta A_{s,u,t} = \int_u^t f(B_r+\psi_s) - f(B_r+\psi_u)\,dr.$$

Hence, by the tower property of conditional expectation and Fubini's Theorem, we get

$$|\mathbb{E}^s\delta A_{s,u,t}| = \Big|\mathbb{E}^s\int_u^t \mathbb{E}^u[f(B_r+\psi_s)-f(B_r+\psi_u)]\,dr\Big|.$$

Now using Lemma 3.3($ii$) with the $\mathcal{F}_u$-measurable variable $\Xi = (\psi_s,\psi_u)$ and using again Fubini's Theorem, we obtain that for $\lambda \in [0,1]$,

$$\begin{aligned}\Big\|\mathbb{E}^s\int_u^t \mathbb{E}^u[f(B_r+\psi_s)-f(B_r+\psi_u)]\,dr\Big\|_{L^q} &\le \int_u^t \|\mathbb{E}^s\|f(\cdot+\psi_s)-f(\cdot+\psi_u)\|_{\mathcal{B}^{\beta-\lambda}_p}\|_{L^q}\,(r-u)^{H(\beta-\lambda-\frac{d}{p})}\,dr\\ &\le C\|f\|_{\mathcal{B}^{\beta-\lambda+1}_p}\,\|\mathbb{E}^s|\psi_u-\psi_s|\|_{L^q}\int_u^t (r-u)^{H(\beta-\lambda-\frac{d}{p})}\,dr.\end{aligned} \tag{A.11}$$

By the conditional Jensen inequality and (2.1) (recall that $m \le q$), we have

$$\|\mathbb{E}^s\,|\psi_u-\psi_s|\|_{L^q} \le [\psi]_{\mathcal{C}^{\tau}_{[s,t]}L^{m,q}}\,(u-s)^{\tau}. \tag{A.12}$$

Then choosing $\lambda = 1$ in (A.11), we get (i$_a$).

In the critical case, let $\tau = 1/2 + H$. For $q = m$, we get from (A.11) and (A.12) that

$$\|\mathbb{E}^s\delta A_{s,u,t}\|_{L^m} \le C\,\|f\|_{\mathcal{B}^{\beta-\lambda+1}_p}\,[\psi]_{\mathcal{C}^{\tau}_{[S,T]}L^m}\,(t-s)^{1+H(\beta-\lambda-\frac{d}{p})+\tau}.$$

Choosing $\lambda = 1$ in the previous inequality, we get (i$'_b$). While choosing $\lambda = 0$ yields (i$_b$).

**Proof of (ii$_a$).** We write

$$\left\|(\mathbb{E}^S|\delta A_{s,u,t}|^m)^{\frac{1}{m}}\right\|_{L^q} \le \left\|(\mathbb{E}^S|\delta A_{s,t}|^m)^{\frac{1}{m}}\right\|_{L^q} + \left\|(\mathbb{E}^S|\delta A_{s,u}|^m)^{\frac{1}{m}}\right\|_{L^q} + \left\|(\mathbb{E}^S|\delta A_{u,t}|^m)^{\frac{1}{m}}\right\|_{L^q},$$

Recall that we already obtained a bound on $\|(\mathbb{E}^S|\delta A_{s,t}|^m)^{1/m}\|_{L^q}$ in (A.10). We obtain similar bounds for $\|(\mathbb{E}^S|\delta A_{s,u}|^m)^{1/m}\|_{L^q}$ and $\|(\mathbb{E}^S|\delta A_{u,t}|^m)^{1/m}\|_{L^q}$, which yields

$$\begin{aligned}\left\|(\mathbb{E}^S|\delta A_{s,u,t}|^m)^{\frac{1}{m}}\right\|_{L^q} &\le C\,\|f\|_{\mathcal{B}_p^\beta}\Big((t-s)^{1+H(\beta-\frac{d}{p})} + (u-s)^{1+H(\beta-\frac{d}{p})} + (t-u)^{1+H(\beta-\frac{d}{p})}\Big)\\ &\le C\,\|f\|_{\mathcal{B}_p^\beta}(t-s)^{1+H(\beta-\frac{d}{p})}.\end{aligned}$$

**Proof of (ii$_b$).** We choose $\varepsilon$ such that $\beta-\varepsilon > -1/2H$ and $\beta-\varepsilon-d/p > -1/H$. We apply now Lemma 3.4 with $\beta \equiv \beta-\varepsilon$ and $\Xi = (\psi_s, \psi_u)$. As $f$ is smooth and bounded, the first assumption of Lemma 3.4 is verified. By Lemma 3.1$(i)$, $\|f(\cdot+\psi_s)-f(\cdot+\psi_u)\|_{\mathcal{B}_p^{\beta-\varepsilon}} \le 2\|f\|_{\mathcal{B}_p^{\beta-\varepsilon}}$, hence the second assumption of Lemma 3.4 is verified. It follows by Lemma 3.4 and Lemma 3.1$(ii)$ that

$$\begin{aligned}\|\delta A_{s,u,t}\|_{L^m} &\le C\,\|\|f(\cdot+\psi_s)-(\cdot+\psi_u)\|_{\mathcal{B}_p^{\beta-\varepsilon}}\|_{L^m}(t-u)^{1+H(\beta-\varepsilon-\frac{d}{p})}\\ &\le C\,\|f\|_{\mathcal{B}_p^\beta}\,\||\psi_s-\psi_u|^\varepsilon\|_{L^m}(t-u)^{1+H(\beta-\varepsilon-\frac{d}{p})}.\end{aligned}$$

Hence by Jensen's inequality,

$$\begin{aligned}\|\delta A_{s,u,t}\|_{L^m} &\le C\,\|f\|_{\mathcal{B}_p^\beta}\|\psi_s-\psi_u\|_{L^m}^\varepsilon(t-u)^{1+H(\beta-\varepsilon-\frac{d}{p})}\\ &\le C\,\|f\|_{\mathcal{B}_p^\beta}[\psi]^\varepsilon_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^m}(t-s)^{1+H(\beta-\frac{d}{p})+\frac{\varepsilon}{2}}\\ &\le C\,\|f\|_{\mathcal{B}_p^\beta}\Big(1+[\psi]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^m}\Big)(t-s)^{1+H(\beta-\frac{d}{p})+\frac{\varepsilon}{2}}.\end{aligned}$$

**Proof of (iii$_a$) and (iii$_b$).** For a sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions of $[S,t]$ with $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$ and mesh size converging to zero, we have

$$\begin{aligned}\Big\|\mathcal{A}_t - \sum_{i=1}^{N_k-1} A_{t_i^k,t_{i+1}^k}\Big\|_{L^m} &\le \sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\|f(B_r+\psi_r)-f(B_r+\psi_{t_i^k})\|_{L^m}\,dr\\ &\le \sum_{i=1}^{N_k-1}\int_{t_i^k}^{t_{i+1}^k}\|f\|_{\mathcal{C}^1}\|\psi_r-\psi_{t_i^k}\|_{L^m}\,dr\\ &\le C\,\|f\|_{\mathcal{C}^1}\,|\Pi_k|^{\tau\wedge(\frac{1}{2}+H)}\,[\psi]_{\mathcal{C}^{\tau\wedge(\frac{1}{2}+H)}_{[s,t]}L^m} \underset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

## A.3 Proof of Proposition 3.9

This proof is very close to the proof of [1, Proposition 7.7], but we adapt it to dimension $d \ge 1$ for the reader's convenience.

Assume w.l.o.g. that $X_0 = 0$ and let $\hat{K} := \hat{X} - \hat{B}$, so that (2.5) is automatically verified. Let $(\tilde{b}^n)_{n\in\mathbb{N}}$ be any sequence of smooth bounded functions converging to $b$ in $\mathcal{B}_p^{\gamma-}$. To verify that $\hat{K}$ and $\hat{X}$ satisfy (2.6), we have to show that

$$\lim_{k\to\infty}\sup_{t\in[0,1]}\left|\int_0^t \tilde{b}^k(\hat{X}_r)\,dr - \hat{K}_t\right| = 0 \text{ in probability.} \tag{A.13}$$

By the triangle inequality we have that for $k, n \in \mathbb{N}$ and $t \in [0,1]$,

$$\begin{aligned}\left|\int_0^t \tilde{b}^k(\hat{X}_r)\,dr - \hat{K}_t\right| \le& \left|\int_0^t \tilde{b}^k(\hat{X}_r)\,dr - \int_0^t \tilde{b}^k(\hat{X}_r^n)\,dr\right| + \left|\int_0^t \tilde{b}^k(\hat{X}_r^n)\,dr - \int_0^t b^n(\hat{X}_r^n)\,dr\right|\\ &+ \left|\int_0^t b^n(\hat{X}_r^n)\,dr - \hat{K}_t\right| =: A_1 + A_2 + A_3.\end{aligned} \tag{A.14}$$

Now we will show that all summands on the right hand side of (A.14) converge to 0 uniformly on $[0,1]$ in probability as $k \to \infty$, choosing $n = n(k)$ accordingly.

First we bound $A_1$. Notice that

$$\begin{aligned}\left|\int_0^t \tilde{b}^k(\hat{X}_r)\,dr - \int_0^t \tilde{b}^k(\hat{X}^n_r)\,dr\right| &\leq \|\tilde{b}^k\|_{\mathcal{C}^1} \int_0^t |\hat{X}_r - \hat{X}^n_r|\,dr \\ &\leq \|\tilde{b}^k\|_{\mathcal{C}^1} \sup_{t\in[0,1]} |\hat{X}_t - \hat{X}^n_t|.\end{aligned}$$

For any $\varepsilon > 0$, choose an increasing sequence $(n(k))_{k\in\mathbb{N}}$ such that

$$\mathbb{P}\Big(\|\tilde{b}^k\|_{\mathcal{C}^1} \sup_{t\in[0,1]} |\hat{X}_t - \hat{X}^{n(k)}_t| > \varepsilon\Big) < \frac{1}{k},\ \forall k \in \mathbb{N}.$$

Hence, we get that

$$\lim_{k\to\infty} \sup_{t\in[0,1]} \left|\int_0^t \tilde{b}^k(\hat{X}_r)\,dr - \int_0^t \tilde{b}^k(\hat{X}^{n(k)}_r)\,dr\right| = 0 \text{ in probability.}$$

Now, we bound $A_2$. Let $\gamma' < \gamma$ with $\gamma' - d/p > 1/2 - 1/(2H)$. By Lemma 3.7 applied to $\hat{X}^n$, $h = \tilde{b}^k - b^n$ and $\gamma'$ instead of $\gamma$, there exists a random variable $Z_{n,k}$ such that

$$\begin{aligned}\mathbb{E}[Z_{n,k}] &\leq C\,\|\tilde{b}^k - b^n\|_{\mathcal{B}^{\gamma'}_p}(1 + \|b^n\|^2_{\mathcal{B}^{\gamma'}_p}) \\ &\leq C\,(\|\tilde{b}^k - b\|_{\mathcal{B}^{\gamma'}_p} + \|b^n - b\|_{\mathcal{B}^{\gamma'}_p})\,(1 + \sup_{m\in\mathbb{N}} \|b^m\|^2_{\mathcal{B}^{\gamma'}_p}),\end{aligned} \tag{A.15}$$

for $C$ independent of $k, n$ and such that there is

$$\sup_{t\in[0,1]} \left|\int_0^t \tilde{b}^k(\hat{X}^n_r)\,dr - \int_0^t b^n(\hat{X}^n_r)\,dr\right| \leq Z_{n,k}.$$

Using Markov's inequality and (A.15) we obtain that

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,1]} \left|\int_0^t \tilde{b}^k(\hat{X}^n_r)\,dr - \int_0^t b^n(\hat{X}^n_r)\,dr\right| > \varepsilon\right) &\leq \varepsilon^{-1}\,\mathbb{E}[Z_{n,k}] \\ &\leq C\,\varepsilon^{-1}\,(\|\tilde{b}^k - b\|_{\mathcal{B}^{\gamma'}_p} + \|b^n - b\|_{\mathcal{B}^{\gamma'}_p})\,(1 + \sup_{m\in\mathbb{N}} \|b^m\|^2_{\mathcal{B}^{\gamma'}_p}).\end{aligned}$$

Choosing $n = n(k)$ as before, we get

$$\lim_{k\to\infty} \sup_{t\in[0,1]} \left|\int_0^t \tilde{b}^k(\hat{X}^{n(k)}_r)\,dr - \int_0^t b^{n(k)}(\hat{X}^{n(k)}_r)\,dr\right| = 0 \text{ in probability.}$$

To bound the last summand $A_3$, recall that $\hat{X}^n_t = \int_0^t b^n(\hat{X}^n_r)\,dr + \hat{B}^n_t$. We get that

$$\sup_{t\in[0,1]} \left|\int_0^t b^n(\hat{X}^n_r)\,dr - \hat{K}_t\right| \leq \sup_{t\in[0,1]} (|\hat{X}^n_t - \hat{X}_t| + |\hat{B}^n_t - \hat{B}_t|).$$

Since by assumption $(\hat{X}^n, \hat{B}^n)_{n\in\mathbb{N}}$ converges to $(\hat{X}, \hat{B})$ on $(\mathcal{C}_{[0,1]})^2$ in probability, we get that

$$\lim_{k\to\infty} \sup_{t\in[0,1]} \left|\int_0^t b^{n(k)}(\hat{X}^{n(k)}_r)\,dr - \hat{K}_t\right| = 0 \text{ in probability,}$$

and therefore (A.13) holds true.

It remains to show that (3.8) holds true. By Lemma 3.6, there exists $C > 0$ such that for any $(s,t) \in \Delta_{0,1}$,

$$\left\|(\mathbb{E}^s|(\hat{X}^n_t - \hat{B}^n_t) - (\hat{X}^n_s - \hat{B}^n_s)|^m)^{\frac{1}{m}}\right\|_{L^\infty} \leq C\,(1 + \sup_{m\in\mathbb{N}} \|b^m\|^2_{\mathcal{B}^\gamma_p})\,(t-s)^{1+H(\gamma - \frac{d}{p})}. \tag{A.16}$$

Using that $\int_0^t b^n(\hat{X}^n_r)\,dr$ converges to $K_t$ on $\mathcal{C}_{[0,1]}$ in probability and that $\sup_{m\in\mathbb{N}} \|b^m\|_{\mathcal{B}^\gamma_p}$ is finite, we get (3.8) by applying Fatou's Lemma to (A.16).

# B Extension of the pathwise uniqueness result

In the regime $\gamma - d/p > 1 - 1/(2H)$, we extend the pathwise uniqueness result of Section 3.5 to weak solutions $X$ that satisfy a weaker regularity than $[X-B]_{\mathcal{C}^{1/2+H}_{[0,1]}L^{2,\infty}} < \infty$. Let $m \geq 2$ and assume that

$$[X-B]_{\mathcal{C}^{H(1-\gamma+\frac{d}{p})+\eta}_{[0,1]}L^{m,\infty}} < \infty \tag{B.1}$$

for some $\eta \in (0,1)$. Our goal is to show that $[X-B]_{\mathcal{C}^{1/2+H}_{[0,1]}L^{m,\infty}} < \infty$. Of course if $\eta \geq H(\gamma - d/p) + 1/2$, this is automatically true. Hence we assume $\eta < H(\gamma - d/p) + 1/2$. Let $n \in \mathbb{N}$ and consider the process

$$X^n_t = X_0 + \int_0^t b^n(X^n_r)\, dr + B_t, \forall t \in [0,1].$$

Applying Proposition 3.5 with $f = b^n$, $\tau = 1 + H(\gamma - d/p)$ and $\psi = X_0 + X^n - B$, we get that there exists a constant $C$ such that for any $n \in \mathbb{N}$, and $(s,t) \in \Delta_{0,1}$,

$$\Big\|\Big(\mathbb{E}^s\Big|\int_s^t b^n(\psi_r + B_r)\, dr\Big|^m\Big)^{\frac{1}{m}}\Big\|_{L^\infty} \leq C\|b\|_{\mathcal{B}^\gamma_p}(t-s)^{1+H(\gamma-\frac{d}{p})} + C[X^n - B]_{\mathcal{C}^{\frac{1}{2}+H}_{[s,t]}L^{m,\infty}}\|b\|_{\mathcal{B}^\gamma_p}(t-s)^{1+H(\gamma-\frac{d}{p})+\tau},$$

where we used that $\|b^n\|_{\mathcal{B}^\gamma_p} \leq \|b\|_{\mathcal{B}^\gamma_p}$. In particular, for $0 \leq S < T \leq 1$ and $(s,t) \in \Delta_{S,T}$, we have

$$\Big\|\Big(\mathbb{E}^s\Big|\int_s^t b^n(\psi_r + B_r)\, dr\Big|^m\Big)^{\frac{1}{m}}\Big\|_{L^\infty} \leq C\|b\|_{\mathcal{B}^\gamma_p}(t-s)^{1+H(\gamma-\frac{d}{p})} + C[X^n - B]_{\mathcal{C}^{\frac{1}{2}+H}_{[S,T]}L^{m,\infty}}\|b\|_{\mathcal{B}^\gamma_p}(t-s)^{1+H(\gamma-\frac{d}{p})+\tau}.$$

Now divide by $(t-s)^{1+H(\gamma-\frac{d}{p})}$ and take the supremum over $\Delta_{S,T}$ to get

$$[X^n - B]_{\mathcal{C}^{1+H(\gamma-\frac{d}{p})}_{[S,T]}L^{m,\infty}} \leq C\,\|b\|_{\mathcal{B}^\gamma_p} + C\,[X^n - B]_{\mathcal{C}^{1+H(\gamma-\frac{d}{p})}_{[S,T]}L^{m,\infty}}\|b\|_{\mathcal{B}^\gamma_p}(t-s)^{\tau-H}, \tag{B.2}$$

with $\tau - H > 1/2$.

For $(s,t) \in \Delta_{S,T}$, we have

$$\Big|\int_s^t b^n(\psi_r + B_r)\, dr\Big| \leq \|b^n\|_{\mathcal{B}^\gamma_p}(t-s) \leq \|b\|_{\mathcal{B}^\gamma_p}(t-s)^{1+H(\gamma-\frac{d}{p})}.$$

Therefore, $[X^n - B]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[S,T]}L^{m,\infty}} < \infty$. Let $\ell = \Big(\frac{1}{2C\|b\|_{\mathcal{B}^\gamma_p}}\Big)^{\frac{1}{1/2-H}}$. Then for $T - S \leq \ell$, (B.2) implies that

$$[X^n - B]_{\mathcal{C}^{1+H(\gamma-\frac{d}{p})}_{[S,T]}L^{m,\infty}} \leq C.$$

Since

$$[X^n - B]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[0,1]}L^{m,\infty}} \leq \sum_{k=0}^{\lfloor\frac{1}{\ell}\rfloor}[X^n - B]_{\mathcal{C}^{1+H(\gamma-d/p)}_{[k\ell,(k+1)\ell]}L^{m,\infty}},$$

and $\ell$ does not depend on $n$, we conclude that

$$\sup_{n\in\mathbb{N}}[X^n - B]_{\mathcal{C}^{1+H(\gamma-\frac{d}{p})}_{[0,1]}L^{m,\infty}} < \infty. \tag{B.3}$$

We now wish to take the limit as $n$ goes to infinity in the previous inequality. Define $K^n := \int_0^\cdot b^n(X_r)\,dr$ and write

$$[X - X^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq [K - K^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + [E^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m}, \tag{B.4}$$

where $K$ is defined in Definition 2.2 and for all $s < t$,

$$E^n_{s,t} := K^n_t - K^n_s - (X^n_t - B_t - X^n_s + B_s) = \int_s^t b^n(X_0 + K_r + B_r) - b^n(X^n_r - B_r + B_r)\,dr.$$

**Bound on $K - K^n$.** For $k \in \mathbb{N}$, we aim to apply Proposition 3.5(a) with $f = b^k - b^n$, $\tau = H(1 - \gamma + d/p) + \eta$, $\beta = \gamma - \eta$ and $\psi = X - B$. Let us check the assumptions: Since $\eta < 1$, we have $\gamma - \eta - d/p > -1/(2H)$; then $\tau$ is clearly positive and $\tau < H + 1/2 < 1$ because we assumed $\eta < H(\gamma - d/p) + 1/2$; finally we have $\tau + H(\gamma - \eta - d/p - 1) = \eta(1 - H) > 0$. In addition, we assumed $[X - B]_{\mathcal{C}^\tau_{[0,1]}L^{m,\infty}} < \infty$ in (B.1), thus Proposition 3.5(a) yields that for any $(s,t) \in \Delta_{S,T}$,

$$\|K^k_t - K^k_s - K^n_t + K^n_s\|_{L^m} \leq C\|b^k - b^n\|_{\mathcal{B}^{\gamma-\eta}_p}(t-s)^{1+H(\gamma-\eta-\frac{d}{p})}.$$

Hence $(K^k_t - K^k_s)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^m(\Omega)$ and therefore it converges. We also know by definition of $X$ that $K^k_t - K^k_s$ converges in probability to $K_t - K_s$. Thus $K^k_t - K^k_s$ converges in $L^m$ to $K_t - K_s$. Now by the convergence of $b^k$ to $b$ in $\mathcal{B}^{\gamma-\eta}_p$, we get

$$\|K_t - K_s - K^n_t + K^n_s\|_{L^m} \leq C\|b - b^n\|_{\mathcal{B}^{\gamma-\eta}_p}\,(t-s)^{1+H(\gamma-\eta-\frac{d}{p})}.$$

Dividing by $(t-s)^{\frac{1}{2}}$ and taking the supremum over $(s,t)$ in $\Delta_{S,T}$ (recall that $\frac{1}{2}+H(\gamma-\eta-d/p) \geq 0$), we get that

$$[K - K^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\|b - b^n\|_{\mathcal{B}^{\gamma-\eta}_p}. \tag{B.5}$$

**Bound on $E^n$.** We apply Proposition 5.2 with $\psi = X^n - B$, $\phi = X_0 + K$, $f = b^n$ and $\tau = \frac{1}{2}$ to get

$$\Big\| \int_s^t b^n(X_0 + K_r + B_r) - b^n(X^n_r - B_r + B_r)\,dr \Big\|_{L^m}$$
$$\leq C\,\|b^n\|_{\mathcal{B}^\gamma_p}(1 + [X^n - B]_{\mathcal{C}^{\frac{1}{2}+H}_{[0,1]}L^m})\Big([X - X^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + \|X_S - X^n_S\|_{L^m}\Big)(t-s)^{1+H(\gamma-1-\frac{d}{p})}.$$

Now divide by $(t-s)^{\frac{1}{2}}$ and take the supremum over $\Delta_{S,T}$ to get

$$[E^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C(1 + [X^n - B]_{\mathcal{C}^{\frac{1}{2}+H}_{[0,1]}L^m})\Big([X - X^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + \|X_S - X^n_S\|_{L^m}\Big)(T-S)^{\frac{1}{2}+H(\gamma-1-\frac{d}{p})}. \tag{B.6}$$

Injecting (B.3), (B.5) and (B.6) into (B.4), we get

$$[X - X^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\|b - b^n\|_{\mathcal{B}^{\gamma-\eta}_p} + C\Big([X - X^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + \|X_S - X^n_S\|_{L^m}\Big)(T-S)^{\frac{1}{2}+H(\gamma-1-\frac{d}{p})}.$$

Hence for $T - S \leq (2C)^{-1/(1/2+H(\gamma-1-d/p))} =: \ell_0$, we get

$$[X - X^n]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq 2C\Big(\|b - b^n\|_{\mathcal{B}^{\gamma-\eta}_p} + \|X_S - X^n_S\|_{L^m}\Big). \tag{B.7}$$

Then the inequality

$$\|X_S - X^n_S\|_{L^m} \leq \|X_{S-\ell_0} - X^n_{S-\ell_0}\|_{L^m} + \ell_0^{\frac{1}{2}}\,[X - X^n]_{\mathcal{C}^{\frac{1}{2}}_{[S-\ell_0,S]}L^m}$$

can be plugged in (B.7) and iterated until $S - k\ell_0$ is smaller than 0 for $k \in \mathbb{N}$ large enough. It follows that

$$[X - X^n]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C \|b - b^n\|_{\mathcal{B}_p^{\gamma-\eta}}.$$

Recall that $b^n$ converges to $b$ in $\mathcal{B}_p^{\gamma-\eta}$ by (2.1). Hence, $X^n$ converges uniformly (in $L^m(\Omega)$) to $X$. Taking the limit as $n$ goes to infinity in (B.3), we have shown that for any $\eta \in (0,1)$ and $m \geq 2$

$$[X - B]_{\mathcal{C}^{H(\gamma-1+\frac{d}{p})+\eta}_{[0,1]} L^{m,\infty}} < \infty \quad \Rightarrow \quad [X - B]_{\mathcal{C}^{1+H(\gamma-\frac{d}{p})}_{[0,1]} L^{m,\infty}} < \infty.$$

Since $1 + H(\gamma - d/p) > 1/2 + H$, we also have $[X - B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{m,\infty}} < \infty$. It follows that pathwise uniqueness holds in the class of weak solutions that satisfy (B.1).

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