Fragility Margin of PWA Control Laws using A Hyperplane Based Binary Search Tree

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Abstract— This paper concentrates on the fragility margins of discrete-time Piecewise Affine (PWA) closed-loop dynamics. Starting from the case where the nominal trajectories are controlled by a PWA controller using a positioning mechanism within a binary search tree (BST), we are interested in preserving the properties of the nominal dynamics, particularly the positive invariance in the presence of perturbations in the control law representation. As the main result, we define (and effectively construct) the fragility margin for a hyperplane defining the partition of the nominal PWA controller. This hyperplane will be identified through a node in a BST, and the proposed margin characterizes the degrees of freedom in the perturbation of its coefficients as it might result from a quantization operation. From the mathematical standpoint, the fragility margin is a set in the coefficients' space, and the properties of this set are formally described.

I. INTRODUCTION

Controller fragility is a primary problem of the performance degradation of feedback control systems due to inaccurate controller implementation [1]. The inaccuracy of the control action is due to different reasons: analog-digital conversion, truncations, quantification, and conversely, resolution of sensing devices and related measurement precision [2]. These negative factors are affecting the controller performances upon implementation. Suppose the analysis of classical linear controllers in terms of fragility is relatively straightforward after passing through the parameter sensitivity tool. By contrast, the robustness analysis of PWA controllers defined in the polyhedral region of partitioned state space is challenging.

The assessment of the robustness of controllers concerning their internal representation requires an in-depth analysis of the design methodologies and the structural implementation details. In recent years, many researchers have investigated the robustness of PWA control laws from different perspectives. For a linear system with PWA control laws, an explicit robustness margin assuming a polytopic uncertainty constraint on the nominal system was proposed in [3]. In [4], the authors extended this work to the direction of admissible variations in the PWA control law coefficients and denoted it as the fragility margin such that the positive invariance of systems states set is guaranteed. In [5], a delay margin based on positive invariance of the nominal closed-loop

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dynamics in the PWA formulation was proposed for PWA systems in the presence of variable time delay. Different from the robustness analysis of the system model and the corresponding PWA controller, in [6], [7], several novel ways of reducing the memory footprint of PWA controller laws were proposed to realize the high accuracy, low precision, and memory efficient embedded model predictive control. In [8], researchers investigated the impact of unavoidable errors on the system performance during the practical application of PWA control laws. In [9], the authors showed that the non-overlapping and the invariance properties of a PWA controller are preserved when the perturbation occurs on the state regions' vertices. However, evaluating the respective control laws in the halfplane representation remains an open problem.

Unlike the vertex-based fragility analysis in [4], [9], [10], this paper focuses on the fragility of the hyperplane characterizing the controller partition using a BST-based approach. In terms of the fragility of the PWA control laws, the main contribution of this work is to establish a margin on the admissible variation of coefficients for a halfspace representation associated with PWA control laws. In detail, we worked on the following aspects:

- The hyperplane-based fragility analysis of the PWA control law is implemented. Although vertex-based fragility in [4], [9], [10] can be used to describe the robustness of the PWA control law in the vertex representation of state partitions, it is challenging to apply. On the one hand, many vertices in a complex system lead to the increased complexity of vertex-based fragility analysis. On the other hand, half-space representations of state partitions are preferred over vertex forms during applications;
- BST is extended to the robustness analysis of the PWA control laws. BST is known to speed up the online point location and reduce the wastage of hardware resources [11]. In this paper, BST is used to overcome the problem of overlapping and region holes in state partition perturbations. In addition to BST, orthogonal search trees are another effective tool for solving such problems; [12]. In this paper, BST is selected for robustness analysis.
- An algorithm is proposed to establish the fragility margin of state partitions with a hyperplane perturbation.

This paper is organized as follows. The section dedicated to the preliminaries presents the necessary knowledge related to PWA. Section III analyzes the equivalent PWA control law based on a BST. The section devoted to the main contribution

states the fragility problem and proposes a fragility margin notion for a PWA control law.

Notation: In this paper, $\mathbb{R}^n$, $\mathbb{R}_+$, $\mathbb{R}_>$ $\mathbb{N}$, $\mathcal{I}_N$ and $\text{card}(\mathcal{I}_N)$ denote the set of real numbers in n-dimensional space, the set of nonnegative real numbers, the set of positive real numbers, the set of nonnegative integers, $\{1, 2, \ldots, N\}$ and the number of elements of the set $\mathcal{I}_N$, respectively. For two column vectors: $x, y \in \mathbb{R}^n$, $x = [x_1, x_2, \ldots, x_n]^T$, $y = [y_1, y_2, \ldots, y_n]^T$, $x \le y$ means $x_i \le y_i$, $\forall i \in \mathcal{I}_n$. $\mathbf{1}_{m \times n}$ ($\mathbf{0}_{m \times n}$), $\mathbf{1}_m$ and $\mathbf{I}_n$ denote a $m \times n$ matrix with its elements equal to one (zero), a column vector with its elements equal to one, and a n-dimensional identity matrix, respectively. For a column vecor $a \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n \times n}$, $\text{diag}(a)$ and $\text{diag}(A)$ denote a $n \times n$ matrix with diagonal vector equal to a, a n dimensional column vector with its elements equal to the value of the diagonal vector of A, respectively. For a matrix $A \in \mathbb{R}^{m \times n}$, $\text{vec}(A)$ denotes the vector composed of the columns of A. For two matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{q \times p}$, their left Kronecker tensor product is defined as:

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix},$$

where a_{ij} represents the element in the i-th row and j-th column of A. For two matrices $A \in \mathbb{R}^{p \times n}$ and $B \in \mathbb{R}^{q \times n}$, the Khatri–Rao product of them denoted by $A \odot B \in \mathbb{R}^{pq \times n} := [A_{\cdot 1} \otimes B_{\cdot 1}, A_{\cdot 2} \otimes B_{\cdot 2}, \cdot, A_{\cdot n} \otimes B_{\cdot n}]$ with $A_{\cdot i}$ and $B_{\cdot i}$ are the i-th column of A and B.

For any vector $x \in \mathbb{R}^n$, we will use throughout the paper the short-hand notation $\bar{x} := \begin{bmatrix} x^T & -1 \end{bmatrix}^T$ to denote the $n+1$ dimensional vector appended with the constant -1.

II. PRELIMINARIES

A. PWA Functions

Definition 1. A hyperplane is a set:

$$\{x \in \mathbb{R}^n | h^T x = b\} = \left\{x \in \mathbb{R}^n \middle| \begin{bmatrix} h \\ b \end{bmatrix}^T \bar{x} = 0\right\}.$$

Definition 2. The function

$$\mathcal{H} : \mathcal{I}_q \to \mathbb{R}^{n+1}$$

defines a collection of q hyperplanes over $\mathcal{X}$ if: $\{x \in \mathbb{R}^n | \mathcal{H}(i)^T \bar{x} = 0\} \cap \mathcal{X} \ne \emptyset$, $\forall i \in \mathcal{I}_q$; $\text{rank}([\mathcal{H}(i), \mathcal{H}(j)]) = 2$, $\forall i, j \in \mathcal{I}_q$ and $i \ne j$.

In the paper, we will use the shorthand symbol to represent the i-th hyperplane within a collection:

$$\mathcal{H}_i := \{x \in \mathbb{R}^n | \mathcal{H}(i)^T \bar{x} = 0\}.$$

Definition 3. Let a collection of hyperplanes be $\mathcal{H}$. Any indicator function $\mathcal{F} : \mathcal{I}_N \times \mathcal{I}_q \to \{-1, 0, 1\}$ is generating

a family of polyhedral sets over $\mathcal{X}$, denoted as[1]

$$\mathcal{R}_i = \{x \in \mathcal{X} | \mathcal{F}(i,j)\mathcal{H}(j)^T \bar{x} \le 0, \forall j \in \mathcal{I}_q\}. \quad (1)$$

Such a family of polyhedral sets is defining a polyhedral partition of $\mathcal{X}$ if $\forall i, j \in \mathcal{I}_N, i \ne j$, there are: $\mathcal{X} = \bigcup_{\forall i \in \mathcal{I}_N} \mathcal{R}_i$; $\mathcal{R}_i \ne \emptyset$; $int(\mathcal{R}_i) \cap int(\mathcal{R}_j) = \emptyset$. This polyhedral partition is denoted next as $\{\mathcal{R}_i\}_{\mathcal{I}_N}$.

Definition 4. Consider a polyhedron $\mathcal{X} \subset \mathbb{R}^n$ with a related polyhedral partition $\{\mathcal{R}_i\}_{\mathcal{I}_N}$. A PWA function over $\{\mathcal{R}_i\}_{\mathcal{I}_N}$ is defined as:

$$f_{pwa} : \left\{ \bigcup_{\forall i \in \mathcal{I}_N} \mathcal{R}_i \right\} \to \mathbb{R}^m$$
$$f_{pwa}(x) = a_i^T x + b_i, \forall x \in \mathcal{R}_i.$$

B. PWA Dynamics

Let a discrete-time PWA system dynamics:

$$x_{k+1} = g_{pwa}(x_k, u_k), \quad (2)$$

with $g_{pwa} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^{n_x}$, where

$$\mathcal{X} = \{x \in \mathbb{R}^{n_x} | H_x x \le b_x\}, \mathcal{U} = \{u \in \mathbb{R}^{n_u} | H_u u \le b_u\},$$

and $H_x \in \mathbb{R}^{n_H \times n_x}$, $b_x \in \mathbb{R}^{n_H}$. One is interested in the PWA system defined on a polyhedral partition $\{\mathcal{S}_s\}_{\mathcal{I}_M}$ of $\mathcal{X}$. The dynamics (2) is denoted as

$$g_{pwa}(x_k, u_k) = A_s x_k + B_s u_k + c_s, x_k \in \mathcal{S}_s$$
$$\mathcal{S}_s = \{x | H_s x \le b_s\}, H_s \in \mathbb{R}^{n_s \times n_x}. \quad (3)$$

C. PWA control

Let a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ associated to a collection of hyperplanes $\mathcal{H}$ and an indicator function $\mathcal{F}$. For the system (2), we consider the controll law as

$$u_k = \kappa_{pwa}(x_k), \forall k \in \mathbb{N}, \quad (4)$$

where $\kappa_{pwa}(x)$ is a PWA function over $\{\mathcal{X}_i\}_{\mathcal{I}_N}$:

$$\kappa_{pwa} : \left\{ \bigcup_{\forall i \in \mathcal{I}_N} \mathcal{X}_i \right\} \to \mathcal{U}, \quad (5)$$
$$\kappa_{pwa}(x) = F_i x + g_i, \text{if } x \in \mathcal{X}_i, i \in \mathcal{I}_N,$$

Based on the control law (5), we obtain the closed loop dynamics of the system (2), denoted as

$$x_{k+1} = g_{pwa}(x_k, \kappa_{pwa}(x_k)) \quad (6)$$

with the explicit formulation

$$g_{pwa}(x_k, \kappa_{pwa}(x_k)) = (A_s + B_s F_i)x_k + B_s g_i + c_s,$$
$$x_k \in \mathcal{S}_s \cap \mathcal{X}_i, s \in \mathcal{I}_M, i \in \mathcal{I}_N.$$

Before discussing the PWA control for the PWA system (2), the definition of a controlled positively invariant set and a relevant assumption should be provided.

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[1]The polyhedron in (1) denotes in a compact formulation the set:

$$\mathcal{R}_i = \left\{ x \in \mathcal{X} \middle| \begin{array}{l} -\mathcal{H}(j)^T \bar{x} \le 0, \text{if } \mathcal{F}(i,j) = -1 \\ \mathcal{H}(j)^T \bar{x} \le 0, \text{if } \mathcal{F}(i,j) = 1 \end{array}, \forall j \in \mathcal{I}_q \right\}.$$

Definition 5. The polyhedral set $\mathcal{X}$ is positively invariant with respect to (w.r.t) the closed system (6) if $\forall x \in \mathcal{X}$, $g_{pwa}(x, \kappa_{pwa}(x)) \in \mathcal{X}$.

Assumption 1. In the current research, we assume that the PWA control law (4) has been designed to guarantee the positive invariance of $\mathcal{X}$ w.r.t (6).

For $\kappa_{pwa}(x)$ satisfying assumption 1, we knew:
- Function $\kappa_{pwa}(x)$ is based on the partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ associated with a known collection of hyperplanes $\mathcal{H}$;
- In order to evaluate $\kappa_{pwa}(x)$, one needs to identify the region $\mathcal{X}_i$ in $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ satisfying $x \in \mathcal{X}_i$. This problem is known as point location, which is effectively addressed by the approach of the so-called BST approach[11];

The robustness/fragility of PWA control laws are related to the BST mechanism. The structural properties inherited from the BST will be analysed in the next section.

III. Binary Search Tree in PWA Control Laws

An effective way to address the point location problem of PWA control laws is to establish an offline BST, which is used online [11]. This section provides the structure and construction of a BST based on a collection of hyperplanes. After illustrating the BST in a real example, one analyzes the features of the PWA control law based on a given BST.

A. Binary Search Tree

The BST is a directional graph denoted as $(\mathcal{R} \cup \mathcal{B} \cup \mathcal{N}, \mathcal{E})$, where $\mathcal{R} \cup \mathcal{B} \cup \mathcal{N}$ represents the set of (root, body, and leaf) nodes of the tree and $\mathcal{E}$ represents the set of edges between two nodes, respectively. The BST contains the following properties:

- Root node $\mathcal{R}$: there is one and only one root node, which has no incoming edge and two outgoing edges;
- Body nodes $\mathcal{B}_i \in \mathcal{B}, i \in \mathcal{I}_{\text{card}(\mathcal{B})}$: each body node $\mathcal{B}_i$ has one incoming edge and two outgoing edges;
- Leaf nodes $\mathcal{N}_i \in \mathcal{N}, i \in \mathcal{I}_{\text{card}(\mathcal{N})}$: each leaf node $\mathcal{N}_i$ has one incoming edge and no outgoing edge;
- Directional edges $\mathcal{E}(n_1, n_2), \mathcal{E}(n_1, n_3) \in \mathcal{E}$, $n_1 \in \mathcal{R} \cup \mathcal{B}$ and $n_2, n_3 \in \mathcal{B} \cup \mathcal{N}$: $\mathcal{E}(n_1, n_2)$ and $\mathcal{E}(n_1, n_3)$ are edges directed from a same node n_1 to two nodes n_2 and n_3. $\mathcal{E}(n_1, n_2)$ and $\mathcal{E}(n_1, n_3)$ characterize the different features of the node n_1, and values 1 and -1 symbolize the difference between them.
- For each leaf node $\mathcal{N}_i, i \in \mathcal{I}_{\text{card}(\mathcal{N})}$, a unique path exists starting from root note $\mathcal{R}$ to $\mathcal{N}_i$, denoted by the edges collection set $\mathcal{P}(\mathcal{N}_i)$ linking $\mathcal{R}$ to $\mathcal{N}_i$.

The construction mechanisms for a BST giving an efficient point location for the PWA control law (5) are out of the scope of this work (the reader is referred to [11]). We are interested in the association between an established BST and the partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ over a collection of hyperplanes $\mathcal{H}$.

- Each node $n_1 \in \mathcal{R} \cup \mathcal{B}$ is associated with a hyperplane in the collection $\mathcal{H}$, and the symbol n_1 represents the index of the selected hyperplane. e.g., hyperplane $\mathcal{H}_i$ is connective with n_1, so $n_1 \mapsto i$;

- Each edge $\mathcal{E}(n_1, n_2)$ is associated with a linear constraint connection with node n_1, and the symbol $\mathcal{E}(n_1, n_2)$ represents a value 1 or -1 to distinguish the different sides of the constraint. e.g., nodes $n_1 \in \mathcal{R} \cup \mathcal{B}$ and $n_2, n_3 \in \mathcal{B} \cup \mathcal{N}$, $\mathcal{E}(n_1, n_2) \mapsto 1$ and $\mathcal{E}(n_1, n_3) \mapsto -1$ are two edges to link them. At the same time, $\mathcal{E}(n_1, n_2)$ denotes constraint $\mathcal{E}(n_1, n_2)\mathcal{H}(n_1)^T \bar{x} \le 0$ while $\mathcal{E}(n_1, n_3)$ denotes constraint $\mathcal{E}(n_1, n_3)\mathcal{H}(n_1)^T \bar{x} \le 0$;
- Each leaf node $\mathcal{N}_i$ is associated to a convex $\mathcal{L}_i \subset \mathcal{X}$ defined by the unique path $\mathcal{P}(\mathcal{N}_i)$. e.g.,

$$\mathcal{L}_i = \bigcap_{\mathcal{E}(n_1, n_2) \in \mathcal{P}(\mathcal{N}_i)} \left\{ x \in \mathcal{X} | \mathcal{E}(n_1, n_2)\mathcal{H}(n_1)^T \bar{x} \le 0 \right\}$$

with $\mathcal{N}_i$ denoting the index of the region $\mathcal{L}_i$. e.g., for $\mathcal{L}_i$, $\mathcal{N}_i \mapsto i$.

According to Definition 3, one can denote a polyhedral partition $\{\mathcal{L}_i\}_{\mathcal{I}_Q}$ by the relevant collection hyperplanes $\mathcal{H}$ and an indicator function $\mathcal{F}_b : \mathcal{I}_Q \times \mathcal{I}_q \to \{-1, 0, 1\}$. The following proposition establishes the link between the indicator function $\mathcal{F}_b$ and the edges of BST.

Proposition 1. For a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ over a collection $\mathcal{H}$ with q hyperplanes and an associated BST $(\mathcal{R} \cup \mathcal{B} \cup \mathcal{N}, \mathcal{E})$, the regions associated with the leaf nodes $\mathcal{N}_i, i \in \mathcal{I}_{\text{card}(\mathcal{N})}$ are described through the indicator function $\mathcal{F}_b(\mathcal{N}_i, n_1)$, which are characterized as

$$\mathcal{L}_i = \{x | \mathcal{F}_b(\mathcal{N}_i, n_1)\mathcal{H}(n_1)^T \bar{x} \le 0, n_1 \in \mathcal{R} \cup \mathcal{B}\}$$

with $\mathcal{F}_b(\mathcal{N}_i, n_1) = \mathcal{E}(n_1, n_2)$ if $\exists n_2 \in \mathcal{B} \cup \mathcal{N}$, s.t. $\mathcal{E}(n_1, n_2) \in \mathcal{P}(\mathcal{N}_i)$, otherwise $\mathcal{F}_b(\mathcal{N}_i, n_1) = 0$, and $\bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i = \bigcup_{i \in \mathcal{I}_Q} \mathcal{L}_i = \mathcal{X}$.

B. Equivalent Controller Based Binary Search Tree

Proposition 2. For a PWA control law (4) with a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, a collection of hyperplanes $\mathcal{H}$ and an indicator function $\mathcal{F}_b : \mathcal{I}_Q \times \mathcal{I}_q \to \{-1, 0, 1\}$ created by an established BST. A BST-based control law is denoted as

$$\kappa_{bst} : \left\{ \bigcup_{\forall i \in \mathcal{I}_Q} \mathcal{L}_i \right\} \to \mathcal{U}, \tag{7}$$
$$\kappa_{bst}(x) = F_j x + g_j, x \in \mathcal{L}_i, j = \tau(\mathcal{L}_i),$$

where $\mathcal{L}_i$ is the i-th region of $\{\mathcal{L}_i\}_{\mathcal{I}_Q}$ and

$$\mathcal{L}_i = \{x \in \mathcal{X} | \mathcal{F}_b(i, j)\mathcal{H}(j)^T \bar{x} \le 0, \forall j \in \mathcal{I}_q\},$$
$$\tau(\mathcal{L}_i) = j, j \in \mathcal{I}_N, \mathcal{L}_i \subset \mathcal{X}_j$$

is an equivalent PWA controller to (4).

Proof. $\forall x \in \mathcal{L}_i, \forall i \in \mathcal{I}_Q$, because $\mathcal{L}_i \subset \mathcal{X}_j$, $\kappa_{bst}(x) = \kappa_{pwa}(x)$. With $\{\mathcal{L}_i\}_{\mathcal{I}_Q}$ a polyhedral partition of $\mathcal{X}$, so $\kappa_{bst}(x) = \kappa_{pwa}(x)$ holds for all $x \in \mathcal{X}$. $\qquad \square$

Remark 1. Based on the equivalent PWA controller (7), the closed loop system (6) can be rewritten as

$$g_{pwa}(x_k, \kappa_{bst}(x_k)) = (A_s + B_s F_j)x_k + B_s g_j + c_s,$$
$$x_k \in \mathcal{S}_s \cap \mathcal{L}_i, s \in \mathcal{I}_M, i \in \mathcal{I}_Q, j = \tau(\mathcal{L}_i).$$

Comparing to the PWA control law (5), there are several features in the equivalent BST-based PWA controller (7):

- The BST-based PWA controller is not unique. Establishing a different BST based on the PWA control law (4), one gets a different equivalent controller over a polyhedral partition with the same collection of hyperplanes;
- The polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ is associated with a sequential research strategy to solve a point location problem, and the partition $\{\mathcal{L}_i\}_{\mathcal{I}_Q}$ is associated with a BST strategy to solve a point location problem;
- If a hyperplane $\mathcal{H}_i, i \in \mathcal{I}_q$ is perturbed to $\tilde{\mathcal{H}}_i := \{x|(\mathcal{H}(i) + \delta_i)^T \bar{x} = 0\}$, these two controllers are no longer equivalent;
- If a hyperplane $\mathcal{H}_i, i \in \mathcal{I}_q$ is perturbed to $\tilde{\mathcal{H}}_i$, the invariant property of the system (2) under the controller (4) and (7) are both affected;
- If a hyperplane $\mathcal{H}_i, i \in \mathcal{I}_q$ is perturbed to $\tilde{\mathcal{H}}_i$, the overlapping and regions hole may exist in the partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, but will not appear in the partition $\{\mathcal{L}_i\}_{\mathcal{I}_Q}$ corresponding to the properties of the BST [11].
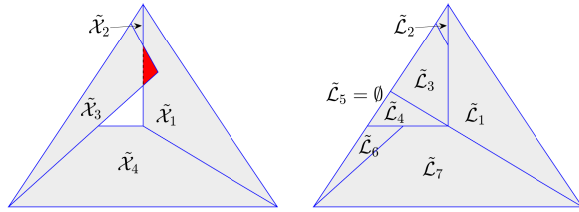
(a) A perturbed polyhedral partition $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_4}$ with overlapping and hole   (b) A perturbed BST-based polyhedral partition $\{\tilde{\mathcal{L}}_i\}_{\mathcal{I}_7}$ without overlapping and hole

Fig. 1.   Polyhedral partition and BST-based polyhedral partition

Figure 1 illustrates these features. Based on these characteristics, one researches the fragility margin of a hyperplane based on a BST in the following sections.

IV. FRAGILITY MARGIN FOR A HYPERPLANE IN THE BST

A. Problem Formulation

Consider a BST-based PWA control law (7) affected by a perturbation on the representation of the hyperplane

$$\tilde{\mathcal{H}}_l := \{x|(\mathcal{H}(l) + \delta_l)^T \bar{x} = 0\}.$$

This perturbation leads to a PWA function:

$$\tilde{\kappa}_{bst} : \left\{ \bigcup_{\forall i \in \mathcal{I}_Q} \tilde{\mathcal{L}}_i^l \right\} \to \mathcal{U}, \tag{8}$$
$$\tilde{\kappa}_{bst}(x_k) = F_j x_k + g_j, \text{if } x_k \in \tilde{\mathcal{L}}_i^l, j = \tau(\mathcal{L}_i),$$

where

$$\tilde{\mathcal{L}}_i^l = \begin{cases} \mathcal{L}_i, & \mathcal{F}_b(i,l) = 0, \\ \mathcal{L}_i^l \cap \mathcal{L}_{i,\delta}^l, & \mathcal{F}_b(i,l) \neq 0, \end{cases} \tag{9}$$

with

$$\mathcal{L}_i^l = \{x \in \mathcal{X}|\mathcal{F}_b(i,j)\mathcal{H}(j)^T \bar{x} \leq 0, \forall j \in \mathcal{I}_q, j \neq l\}$$
$$= \{x|\bar{H}_i x \leq \bar{b}_i\}, \bar{H}_i \in \mathbb{R}^{\bar{n}_i \times n_x}, \bar{b}_i \in \mathbb{R}^{\bar{n}_i},$$
$$\mathcal{L}_{i,\delta}^l = \{x|\mathcal{F}_b(i,l)(\mathcal{H}(l) + \delta_l)^T \bar{x} \leq 0\}.$$

For the sake of brevity, let us express $\mathcal{L}_i^l \cap \mathcal{L}_{i,\delta}^l$ explicitly as

$$\left\{ x \left| \begin{bmatrix} \mathcal{F}_b(i,l)(h_l + \delta_l^h)^T \\ \bar{H}_i \end{bmatrix} x \leq \begin{bmatrix} \mathcal{F}_b(i,l)(b_l + \delta_l^b) \\ \bar{b}_i \end{bmatrix} \right. \right\}, \tag{10}$$

where $h_l, \delta_l^h \in \mathbb{R}^{n_x}, b_l, \delta_l^b \in \mathbb{R}, \begin{bmatrix} h_l \\ b_l \end{bmatrix} = \mathcal{H}(l)$ and $\begin{bmatrix} \delta_l^h \\ \delta_l^b \end{bmatrix} = \delta_l$.

The goal is to find a fragility margin for the perturbation of the hyperplane $\tilde{\mathcal{H}}_l$. This margin is characterized in terms of a region $\Delta_l \subset \mathbb{R}^{n_x+1}$ such that $\forall \delta_l \in \Delta_l$, the positive invariance of $\mathcal{X}$ is ensured under the perturbed PWA control law (8). In other words, $\forall x \in \mathcal{S}_s \cap \tilde{\mathcal{L}}_i^l, i \in \mathcal{I}_Q, s \in \mathcal{I}_M$:

$$\Delta_l = \{\delta_l \in \mathbb{R}^{n_x+1}| \text{ s.t. (11)}\},$$
$$(A_s + B_s F_j)x + B_s g_j + c_s \in \mathcal{X}, j = \tau(\mathcal{L}_i). \tag{11}$$

Proposition 3. The fragility margin Δ_l is represented as

$$\Delta_l = \bigcap_{\forall i \in \mathscr{I}_l} \bigcap_{\forall s \in \mathcal{I}_M} \Delta_{l,i}^s$$

with

$$\mathscr{I}_l = \{i|\forall i \in \mathcal{I}_Q, \mathcal{F}_b(i,l) \neq 0\},$$
$$\Delta_{l,i}^s = \left\{\delta_l \in \mathbb{R}^{n_x+1}|\forall x \in \mathcal{S}_s \cap \tilde{\mathcal{L}}_i^l, \text{ s.t. (11)}\right\}. \tag{12}$$

Proof. Based on the assumption 1 and (9), when $\forall i \in \mathcal{I}_Q \setminus \mathscr{I}_l, \tilde{\mathcal{L}}_i^l = \mathcal{L}_i$ and condition (11) holds directly. So for the particular indexes $i \in \mathscr{I}_l$, we can construct the fragility margin Δ_l as an intersection of local margins $\Delta_{l,i}^s$ for the s-th local dynamic with $s \in \mathcal{I}_M$. ☐

Remark 2. The particular BST construction detailed in the previous sections identifies the local margins relevant for the construction and thus avoids the treatment of each region composing the partition of the closed loop PWA. The following subsection concentrates on the practical computation of the local margin $\Delta_{l,i}^s$.

B. Construction of the Fragility Margin

Unlike the fragility margin focused on controller parameters defined in [4], the present fragility margin deals with the practical evaluation of the PWA control and thus is particularly relevant for control law implementation. While both notions have a similar construction process, the hyperplane fragility will prove to deserve a series of technological transformations for effective computation.

Proposition 4. Consider a linear PWA system (2) with a collection of hyperplanes $\mathcal{H}$ and a perturbed PWA control law (8). The local fragility margin w.r.t the hyperplane $\mathcal{H}_l$ is obtained as

$$\Delta_{l,i}^s = \text{Proj}_{(\delta_l^h, \delta_l^b)} \mathscr{O}_{l,i}^s.$$

Set $\mathscr{O}_{l,i}^s$ represents a polyhedron

$$\mathscr{O}_{l,i}^s = \{(\delta_l^h, \delta_l^b, \Upsilon_1, \Upsilon_2)|$$

$$[\mathbf{1}_{n_H}, \Upsilon_2]\begin{bmatrix} \mathcal{F}_b(i,l)(h_l + \delta_l^h)^T \\ \bar{H}_i \\ H_s \end{bmatrix} = \Upsilon_1 H_x(A_s + B_s F_j),$$

$$[\mathbf{1}_{n_H}, \Upsilon_2]\begin{bmatrix} \mathcal{F}_b(i,l)(b_l + \delta_l^b) \\ \bar{b}_i \\ b_s \end{bmatrix} \leq \Upsilon_1(b_x - H_x(B_s g_j + c_s))\}$$

$$(13)$$

with $(\delta_l^h, \delta_l^b, \Upsilon_1, \Upsilon_2) \in \mathbb{R}^{n_x} \times \mathbb{R} \times \mathbb{R}_{>}^{n_H \times n_H} \times \mathbb{R}_{+}^{n_H \times (\bar{n}_i + n_s)}$ and Υ_1 is a diagonal matrix.

Proof. First, let us note that $\mathcal{F}_b(i,l) \in \{-1, 0, 1\}$ from the definition and $\mathcal{F}_b(i,l) \neq 0$. Thus, to simplify, without loss of generality, we consider the case $\mathcal{F}_b(i,l) = 1$. For $i \in \mathscr{I}_l$ and $\forall x \in \mathcal{S}_s \cap \tilde{\mathcal{L}}_i^l$, in order to hold the invariance property, the following should hold:

$$(A_s + B_s F_j)x + B_s g_j + c_s \in \mathcal{X}, j = \tau(\mathcal{L}_i).$$

From the halfspace representation of the state set $\mathcal{X}$, it can be seen that $\forall x \in \mathcal{S}_s \cap \tilde{\mathcal{L}}_i^l \subset \mathcal{X}$ and thus:

$$H_x((A_s + B_s F_j)x + B_s g_j + c_s) \leq b_x.$$

In other word, it means that $\mathcal{S}_s \cap \tilde{\mathcal{L}}_i^l \subset \mathscr{H}_{s,i}$, where

$$\mathscr{H}_{s,i} = \{x | H_x(A_s + B_s F_j)x \leq b_x - H_x(B_s g_j + c_s)\}.$$

Combining (3) and (10), $\mathcal{S}_s \cap \tilde{\mathcal{L}}_i^l$ can be written as:

$$\left\{ x \middle| \begin{bmatrix} h_l^T + (\delta_l^h)^T \\ \bar{H}_i \\ H_s \end{bmatrix} x \leq \begin{bmatrix} b_l + \delta_l^b \\ \bar{b}_i \\ b_s \end{bmatrix} \right\}.$$

According to the Extended Fakas Lemma [13], if there exist two matrices $\Gamma_1 \in \mathbb{R}_+^{n_H}$ and $\Gamma_2 \in \mathbb{R}_+^{n_H \times (\bar{n}_i + n_s)}$ such that:

$$[\Gamma_1 \quad \Gamma_2]\begin{bmatrix} h_l^T + (\delta_l^h)^T \\ \bar{H}_i \\ H_s \end{bmatrix} = H_x(A_s + B_s F_j), \quad (14a)$$

$$[\Gamma_1 \quad \Gamma_2]\begin{bmatrix} b_l + \delta_l^b \\ \bar{b}_i \\ b_s \end{bmatrix} \leq b_x - H_x(B_s g_j + c_s). \quad (14b)$$

then $\mathcal{S}_s \cap \tilde{\mathcal{L}}_i^l \subset \mathscr{H}_{s,i}$. Formulations (14) give the nonlinear constraint relation between $\delta_l^h, \delta_l^b, \Gamma_1$ and Γ_2. Indeed, the nonlinear terms $\Gamma_1(\delta_l^h)^T$ and $\Gamma_1\delta_l^b$ are challenging to handle, and thus it is not easy to obtain a feasible set for (δ_l^h, δ_l^b) from (14). In order to get a feasible sets for (δ_l^h, δ_l^b), further processing is required.

The domain $\mathbb{R}_+^{n_H}$ of definition of Γ_1 is equivalent to

$$\mathbb{R}_+^{n_H} \iff \bigcup_{\forall \mathscr{G}_1 \in \mathscr{G}} \Psi(\mathscr{G}_1) \quad (15)$$

with $\mathscr{G}$ the set of all subsets of $\mathcal{I}_{n_H}$ and

$$\Psi(\mathscr{G}_1) = \{x | e_m^T x = 0, e_n^T x > 0, \forall m \in \mathscr{G}_1, n \in \mathcal{I}_{n_H} \setminus \mathscr{G}_1\}$$

where $x \in \mathbb{R}^{n_H}$, $e_m, e_n \in \mathbb{R}^{n_H}$ are vectors with the m-th and n-th element 1 and other elements 0. So the feasible set for (δ_l^h, δ_l^b) satisfying $\Gamma_1 \in \mathbb{R}_+^{n_H}$ can be composed as:

$$\Delta_{l,i}^s(\mathbb{R}_+^{n_H}) = \bigcup_{\forall \mathscr{G}_1 \in \mathscr{G}} \Delta_{l,i}^s(\Psi(\mathscr{G}_1)),$$

where $\Delta_{l,i}^s(\Psi(\mathscr{G}_1))$ is a feasible set for (δ_l^h, δ_l^b) under the constraint $\Gamma_1 \in \Psi(\mathscr{G}_1)$.

For the formulas (14) under the constraint $\Gamma_1 \in \Psi(\mathscr{G}_1)$, there exists another two matrices $\Upsilon_1 \in \mathbb{R}_{>}^{n_H}$ and $\Upsilon_2 \in \mathbb{R}_+^{n_H \times (\bar{n}_i + n_s)}$ such that:

- $\Upsilon_1 \in \mathbb{R}_{>}^{n_H \times n_H}$ is a positive diagonal matrix, $\Upsilon_1\Gamma_1 = \mathbf{O}_{n_H}$ with $e_m^T \mathbf{O}_{n_H} = 0$ and $e_n^T \mathbf{O}_{n_H} = 1$ for all $m \in \mathscr{G}_1$ and $n \in \mathcal{I}_{n_H} \setminus \mathscr{G}_1$;
- $\Upsilon_2 = \Upsilon_1\Gamma_2$.

Both sides of (14) are left multiplied by a matrix Υ_1,

$$[\mathbf{O}_{n_H} \quad \Upsilon_2]\begin{bmatrix} h_l^T + (\delta_l^h)^T \\ \bar{H}_i \\ H_s \end{bmatrix} = \Upsilon_1 H_x(A_s + B_s F_j), \quad (16)$$

$$[\mathbf{O}_{n_H} \quad \Upsilon_2]\begin{bmatrix} b_l + \delta_l^b \\ \bar{b}_i \\ b_s \end{bmatrix} \leq \Upsilon_1(b_x - H_x(B_s g_j + c_s)). \quad (17)$$

These are linear constraints for variables

$$(\delta_l^h, \delta_l^b, \Upsilon_1, \Upsilon_2) \in \mathbb{R}^{n_x} \times \mathbb{R} \times \mathbb{R}_{>}^{n_H} \times \mathbb{R}_+^{n_H \times (\bar{n}_i + n_s)},$$

which can be denoted as a convex set:

$$\mathscr{O}_{l,i}^s(\Psi(\mathscr{G}_1)) = \{(\delta_l^h, \delta_l^b, \Upsilon_1, \Upsilon_2)| \text{ s.t. } (16), (17)\}.$$

Thus, $\Delta_{l,i}^s(\Psi(\mathscr{G}_1)) = \text{Proj}_{(\delta_l^h, \delta_l^b)} \mathscr{O}_{l,i}^s(\Psi(\mathscr{G}_1))$.

Combing with (15), the feasible set $\Delta_{l,i}^s(\mathbb{R}_+^{n_H})$ is a union of convex sets, which is generally not a convex set. Any subset of the feasible set $\Delta_{l,i}^s(\mathbb{R}_+^{n_H})$ can be considered as a local fragility margin w.r.t the hyperplane $\mathcal{H}_l$, so we achieve a proof of the proposition as soon as we can prove $\Delta_{l,i}^s \subset \Delta_{l,i}^s(\mathbb{R}_+^{n_H})$.

If one assumes $\Gamma_1 \in \Psi(\mathscr{G}_1)$ and $\mathscr{G}_1 = \emptyset$, then: $\Psi(\mathscr{G}_1) = \mathbb{R}_{>}^{n_H}$; $\Upsilon_1 \in \mathbb{R}_{>}^{n_H \times n_H}$ is a positive diagonal matrix, and $\Upsilon_1 = \text{diag}(\Gamma_1)^{-1}$; $\mathbf{O}_{n_H} = \mathbf{1}_{n_H}$.

It means $\mathscr{O}_{l,i}^s = \mathscr{O}_{l,i}^s(\emptyset)$. Because $\Delta_{l,i}^s = \text{Proj}_{(\delta_l^h, \delta_l^b)} \mathscr{O}_{l,i}^s$, $\Delta_{l,i}^s = \Delta_{l,i}^s(\emptyset) \subset \Delta_{l,i}^s(\mathbb{R}_+^{n_H})$. $\square$

C. Computational Aspects and Relevant Algorithm

In this subsection, in order to compute the fragility margin Δ_l, an equivalent canonical representation is proposed.

Actually, according to the vectorized operator and the properties of the Kronecker tensor product, one applies a vectorized operator to both sides of the formulation (13), and the result is:

$$Ez = E_0, Qz \leq Q_0$$

with

$$z = \begin{bmatrix} (\delta_l^h)^T & \delta_l^b & \text{diag}(\Upsilon_1)^T & \text{vec}(\Upsilon_2)^T \end{bmatrix}^T,$$
$$E = \begin{bmatrix} E_1 & E_2 & E_3 & E_4 \end{bmatrix}, Q = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \end{bmatrix},$$
$$E_0 = h_l \otimes \mathbf{I}_{n_H}, E_1 = \mathbf{I}_{n_x} \otimes \mathbf{1}_{n_H}, E_2 = \mathbf{0}_{n_x \times n_H},$$
$$Q_0 = b_l \otimes \mathbf{I}_{n_H}, Q_1 = \mathbf{0}_{n_H \times n_x}, Q_2 = \mathbf{I}_{n_H},$$
$$E_3 = -(H(A_s + B_s F_j))^T \otimes \mathbf{I}_{n_H}(\mathbf{I}_{n_H} \odot \mathbf{I}_{n_H}),$$
$$Q_3 = -(b_x - H_x B_s g_j - H_x c_s)^T \otimes \mathbf{I}_{n_H}(\mathbf{I}_{n_H} \odot \mathbf{I}_{n_H}),$$
$$E_4 = \begin{bmatrix} \bar{H}_i \\ H_s \end{bmatrix}^T \otimes \mathbf{I}_{n_H}, Q_4 = \begin{bmatrix} \bar{b}_i \\ b_s \end{bmatrix}^T \otimes \mathbf{I}_{n_H}.$$

As the nonnegative constraints on Υ_1 and Υ_2, one can rewrite them as $M_1 z < M_{10}, M_2 z \leq M_{20}$, with $M_{10} = \mathbf{0}_{n_H}$, $M_{20} = \mathbf{0}_{H_l}$ and

$$M_1 = \begin{bmatrix} \mathbf{0}_{n_H \times n_x} & \mathbf{0}_{n_H} & -\mathbf{I}_{n_H} & \mathbf{0}_{n_H \times H_l} \end{bmatrix},$$
$$M_2 = \begin{bmatrix} \mathbf{0}_{H_l \times n_x} & \mathbf{0}_{H_l} & \mathbf{0}_{H_l \times n_H} & -\mathbf{I}_{H_l} \end{bmatrix},$$

where $H_l = n_H(\bar{n}_i + n_s)$. So the set $\mathscr{O}_{l,i}^s$ can be transformed into a polytope formulation as: $\mathscr{O}_{l,i}^s \subset \mathbf{R}^{n_x+1+n_H+H_l}$ and

$$\mathscr{O}_{l,i}^s = \{z | Ex = E_0, Qz \leq Q_0, M_1 z < M_{10}, M_2 z \leq M_{20}\}.$$

Therefore, the local fragility margin of a hyperplane $\mathcal{H}_l \in \mathcal{H}$ is finally calculated as: $\Delta_{l,i}^s = \text{Proj}_{\delta_l^h, \delta_l^b} \mathscr{O}_{l,i}^s$.

Remark 3. If $\mathcal{F}_s(i,l) = -1$, the related fragility margin $\Delta_{l,i}^s$ should be calculated according to Proposition 4 with adjusting the relevant formulation sign.

Algorithm 1 Fragility margin for a hyperplane $\mathcal{H}_l \in \mathcal{H}$

Require: $x_{k+1} = g_{pwa}(x_k, \kappa_{pwa}(x_k))$, a closed loop PWA system in (6), the polyhedral partitions $\{\mathcal{S}_s\}_{\mathcal{I}_M}$ and $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ over the state set $\mathcal{X}$;

Ensure: The fragility margin of the hyperplane $\mathcal{H}_l$;
1: Construction a collection of hyperplanes $\mathcal{H}$ for $\{\mathcal{X}_i\}_{\mathcal{I}_N}$;
2: Establishing a BST for the PWA control law $u_k = \kappa_{pwa}(x_k)$ based on the algorithm 2 in [11];
3: Recording an equivalent controller (7) with a BST-based polyhedral partition $\{\mathcal{L}_i\}_{\mathcal{I}_Q}$;
4: Given a hyperplane $\mathcal{H}_l \in \mathcal{H}$, getting the index set $\mathscr{I}_l = \{i | \forall i \in \mathcal{I}_Q, \mathcal{F}_b(i,l) \neq 0\}$, initial $i \in \mathscr{I}_l$ and $\Delta_l = \mathbb{R}^{n_x+1}$;
5: **while** $\mathscr{I}_l \neq \emptyset$ **do**
6:     **for** s=1:M **do**
7:         Calculating $\Delta_{l,i}^s$ based on Proposition 4;
8:         $\Delta_l = \Delta_l \cap \Delta_{l,i}^s$;
9:     **end for**
10:    $\mathscr{I}_l = \mathscr{I}_l \setminus i$;
11:    Selecting $i \in \mathscr{I}_l$;
12: **end while**
13: **return** A hyperplane $\mathcal{H}_l$ and its fragility margin Δ_l.

For the closed loop PWA system (6), Algorithm 1 shows the process of calculating the fragility margin for a given hyperplane based on the former content. In the worse case, $\text{card}(\mathscr{I}_l) = Q$, means that one must calculate $Q * M$ fragility margin sets and $Q * M$ intersection operation.

V. EXAMPLE

Consider a discrete-time PWA system:

$$x_{k+1} = g_{pwa}(x_k, u_k) \tag{18}$$

with the PWA function g_{pwa}, the states and inputs constraints

$$g_{pwa}(x_k, u_k) = \begin{cases} A_1 x_k + B_1 u_k, [0,1]x_k \leq 0 \\ A_2 x_k + B_2 u_k, [0,1]x_k \geq 0 \end{cases},$$
$$\begin{bmatrix} -5 \\ -10 \end{bmatrix} \leq x_k \leq \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \|u_k\|_\infty \leq 5, \forall k \in \mathbb{N}.$$

The system parameter matrices are:

$$A_1 = \begin{smallmatrix} 1.40 & 0 \\ 1.80 & -1.10 \end{smallmatrix}, A_2 = \begin{bmatrix} 1.22 & -0.08 \\ 0.78 & -1.26 \end{bmatrix},$$
$$B_1 = B_2 = \begin{bmatrix} 0.50 \\ 0.70 \end{bmatrix}.$$

We design an explicit model predictive controller for an auxiliary linear system $x_{k+1} = A_1 x_k + B_1 u_k$ with the same parameters and constraints to the system (18). This problem has parameters $Q = \mathcal{I}_2$, $R = 1$, a prediction step equals 2, an LQR terminal set, and a terminal state cost

$$P = \begin{bmatrix} 5.6902 & 0.4862 \\ 0.4862 & 3.1099 \end{bmatrix}.$$

Based on these parameters, a PWA controller with 13 linear-affine control laws and their associated state space partitions can be obtained. For this PWA controller in a closed loop with the system (18), the positive invariance property is verified over the definition domain. In the following simulation, using the presented procedure, one can construct the fragility margin for such a PWA controller.

A BST was constructed according to the algorithm in [11]. In current example, one obtains a BST with depth 6, $\{\mathcal{L}_i\}_{\mathcal{I}_{27}}$ and a collection $\mathcal{H}$ with 28 hyperplanes. Following the procedure of Algorithm 1, a perturbed hyperplane $\mathcal{H}_l$ with $h_l = [-0.9795 \quad 0.2013]^T$ and $b_l = 1.9359$ was considered. Correspondingly, the index set $\mathscr{I}_l$ was established with $\text{card}(\mathscr{I}_l) = 4$.

In order to illustrate the effectiveness of the proposed fragility margin, three simulation scenarios were shown:
- Scenario 1: the fragility margin Δ_l of $\mathcal{H}_l$ was exhibited;
- Scenario 2: a random element $[(\delta_{l,in}^h)^T, \delta_{l,in}^b]^T \in \Delta_l$ was selected to verify the positive invariant property been hold after perturbation.
- Scenario 3: an element $[(\delta_{l,out}^h)^T, \delta_{l,out}^b]^T \notin \Delta_l$ was selected to verify the positive invariant property not been hold after perturbation.

Scenario 1: Figure 2 depicts the selected hyperplane and the regions affected by the perturbation on the coefficients. The fragility margin Δ_l of hyperplane $\mathcal{H}_l$ is illustrated in Figure 3. It should be pointed out that the actual margin calculated through algorithm 1 may be unbounded. It is related to the equivalence of representation up to the multiplication with positive constants. In order to preserve the readability, a bounding box $\|x\|_\infty \leq 10$ was considered for the parameters of the hyperplane.
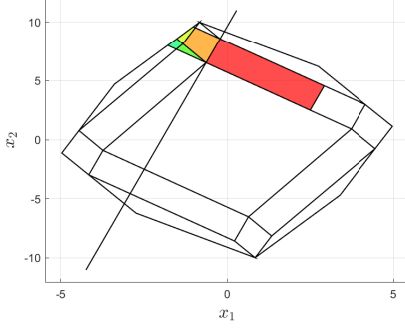
Fig. 2. Selected hyperplane $\mathcal{H}_l$ and relevant affected regions.
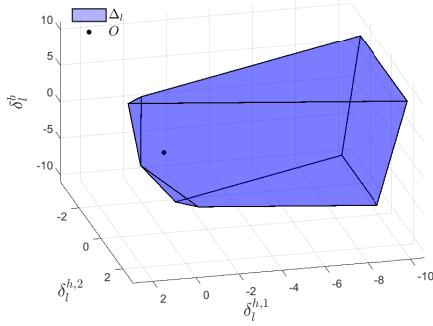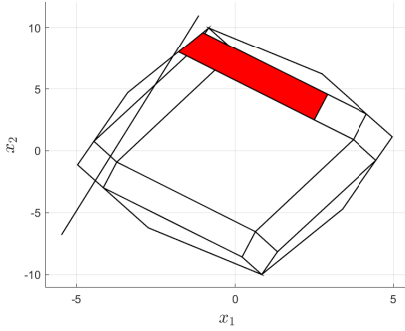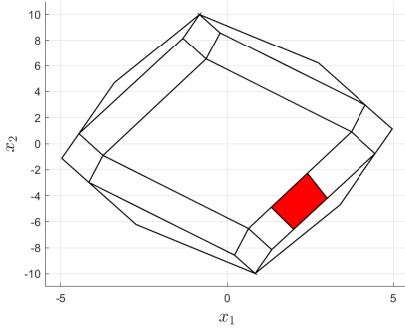

Fig. 3. The fragility margin Δ_l of the hyperplane $\mathcal{H}_l$.

Scenario 2: A random selection of 2565 points in the feasible set has been performed. The property under investigation is the positive invariance of the state regions, which holds despite the perturbation on the selected hyperplane. We select a point $[(\delta_{l,in}^h)^T, \delta_{l,in}^b]^T = [0.4726 \;\; -0.0779 \;\; 0]^T$ to illustrate the effectiveness of the admissible perturbations (see Figure 4).

The subgraph (a) of Figure 4 shows the perturbed regions, while the subgraph (b) of Figure 4 shows the invariant property of the perturbed system. The invariance property is guaranteed if the state of these affected regions remains within the domain of definition in one step. In subgraph (b), the regions with color are the one-step image of the regions with the same color in subgraph (a) under the action of the PWA controller. Intuitively, these regions are subsets of the whole state region, so the invariance property holds under the perturbation.

Scenario 3: In contrast, a point $[(\delta_{l,out}^h)^T, \delta_{l,out}^b]^T \notin \Delta_l$ equaling to $[-0.2628 \;\; 0.3523 \;\; 0.3250]^T$ is selected to illustrate the invalidation of invariant property. Similarly, one shows the perturbed regions in subgraph (a) of Figure 5 and the invariant property invalidation in subgraph (b) of Figure 5. As it can be observed graphically, the yellow region is not a subset of the original definition set, and the positive invariance is lost for this perturbation of parameters.

(a) The perturbation hyperplane and related regions

(a) The perturbation hyperplane and related regions

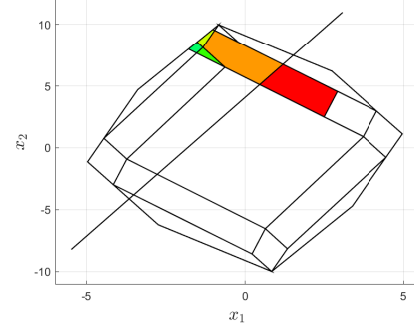
(b) The one-step ahead image of the regions affected by the perturbation.

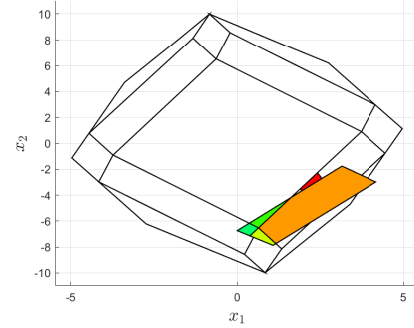Fig. 4. A perturbation hyperplane in its fragility margin.

(b) The one-step ahead image of the regions affected by the perturbation.

Fig. 5. A perturbation hyperplane out of its fragility margin.

VI. CONCLUSIONS

A fragility margin of the hyperplane in PWA control is proposed by incorporating the impact of the BST into its practical assessment. The technique focuses on the robustness of the regions associated with a PWA control law. The positive invariance of the closed-loop PWA system is guaranteed if the perturbation hyperplane that contributes to the definition of the polyhedral partition is confined to its fragility margin. A fragility margin independent of a particular BST configuration shall be investigated further.

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