Uniform Global Asymptotic Stability for Time-Invariant Delay Systems

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Abstract— **For time-invariant finite-dimensional systems, it is known that global asymptotic stability (GAS) is equivalent to uniform global asymptotic stability (UGAS), in which the decay rate and transient overshoot of solutions are requested to be uniform on bounded sets of initial states. This paper investigates this relationship for time-invariant delay systems. We show that UGAS and GAS are equivalent for this class of systems under the assumption of robust forward completeness, i.e. under the assumption that the reachable set from any bounded set of initial states on any finite time horizon is bounded. We also show that, if the state space is a space in a particular family of Sobolev or Hölder spaces, then GAS is equivalent to UGAS and that robust forward completeness holds. Based on these equivalences, we provide a novel Lyapunov characterization of GAS (and UGAS) in the aforementioned spaces.**

I. Introduction

For time-invariant, finite-dimensional systems described by ordinary differential equations, GAS is traditionally defined as the combination of Lyapunov stability and global convergence of solutions to the origin. An alternative way to state it is through a $\mathcal{KL}$ bound on the solutions' norm. This alternative description is seemingly more demanding than merely stability and global attractiveness as it additionally imposes that the convergence rate and the transient overshoot of solutions are uniform over bounded sets of initial states, thus leading to the notion of Uniform Global Asymptotic Stability (UGAS). This extra conservatism turns out to be only apparent: it is well known that, for such systems, GAS and UGAS are actually equivalent properties [24], [29].

The importance of this uniformity is twofold. First, from a practical perspective, it rules out the possibility of having an arbitrarily slow convergence of solutions to the origin or an arbitrarily large transient overshoot when initial states are confined to a bounded set. Second, it constitutes a key requirement for the construction of Lyapunov functions and is at the basis of important stability properties for systems with inputs such as Input-to-State Stability [25] and Input-to-Output Stability [27]. To that respect, it is worth mentioning that this uniformity no longer comes for free when considering output stability properties [21], [10].

Another important feature of time-invariant finite-dimensional systems is that the existence of their solutions for all positive times (forward completeness) ensures a bounded reachable set over any finite time horizon from every bounded set of initial conditions [18], [26]. In other words, starting from a bounded set of initial states, the solutions of a time-invariant finite-dimensional system remain bounded over a finite time horizon. In the literature, this property is referred to as either Robust Forward Completeness (RFC) [11] or bounded reachable sets property [19] and plays a crucial role in the Lyapunov characterization of forward completeness [1].

For general infinite-dimensional systems, the equivalence between GAS and UGAS and between forward completeness and RFC is far more delicate. In particular, an example is given in [20] of an infinite-dimensional system which is forward complete yet not RFC. Nevertheless, it is still an open question whether such equivalences hold when considering only time-delay systems.

Partial answers do exist though. For systems described by neutral functional differential equations, the equivalence between GAS and UGAS does not hold, even in the linear time-invariant case: see Lemma 1.1 and Example 1.6 in [9]. For time-delay systems, the relationship between GAS and UGAS was recently discussed in [23]. Interestingly, as far as local properties are concerned, asymptotic stability is indeed equivalent to uniform asymptotic stability for time-invariant delay systems [8, Lemma 1.1, p. 131]. This local result was actually proved five decades ago for globally Lipschitz time-delay systems (see Condition 4 on page 128, Definition 28.1 on page 131, Definition 30.2 on page 146, and pages 150-151 in [17]). In [16, Theorem 6.3.1, p. 73], it is stated that GAS is equivalent to UGAS for periodic delay systems provided that the function describing the dynamics is Lipschitz on bounded sets. The proof is not provided in [15] but [17] is quoted for. However, as explained above, the results and the proofs provided in [17] do not show the equivalence between GAS and UGAS for the class of systems considered in [15].

Thus, it is not known whether GAS and UGAS are equivalent for delay systems. Similarly, it is not known whether forward completeness is equivalent to RFC for such systems, the consequences of which are discussed in [19].

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Although not fully solving them, the present note shows that these two open questions are related. More specifically we show that, under the assumption of RFC, GAS and UGAS are indeed equivalent properties for time-invariant delay systems (Theorem 1). Since RFC holds automatically for globally Lipschitz delay systems and can often be established using Lyapunov techniques, our result constitutes a significant generalization of the result in [17]. The second contribution of this note is to show that the answer to both open questions depends crucially on the selection of the state space. More precisely, we show that if the considered state space is the Sobolev space $W^{1,p}([-r,0])$ with $p \in (1,+\infty]$, then, under a forward completeness assumption, GAS is indeed equivalent to UGAS and RFC holds for time-invariant delay systems, just like the finite-dimensional case (Theorems 2 and 3). Similarly, we show that if the considered state space is the Hölder space $C^{0,1-1/p}$ with $p \in (1,+\infty]$, then, under the usual forward completeness assumption, GAS is also equivalent to UGAS and RFC holds (Theorems 2 and 3). Here, it should be emphasized that Sobolev spaces have already been used as state spaces in the literature for neutral delay systems [6], [16]. The third contribution of the paper exploits this equivalence to propose a novel Lyapunov characterization of GAS (hence, UGAS) for time-invariant delay systems when treating the state space as one of the aforementioned spaces (Theorem 5 for $W^{1,p}([-r,0])$ or $C^{0,1-1/p}([-r,0])$ with $p \in (1,+\infty]$ and Theorem 6 for $W^{1,p}([-r,0])$ with $p \in (1,+\infty)$).

The consequences of the obtained results to control theory are important. The equivalence of GAS and UGAS allows the control designer to use tools for feedback design that can prove global attractivity but not global uniform attractivity and still argue that UGAS holds. Such tools include the extension of LaSalle's theorem to delay systems and Barbălat's lemma (see for instance [7], [10]). Another important issue for control theory is robustness to various external inputs (disturbances). It has been shown that robustness to persistent external inputs is a consequence of uniform stability notions (see the discussion on page 162 in [17] for the case of delay systems as well as the discussion in [28] for the finite-dimensional case). The results of the present work allow the control designer to be sure that a feedback law which induces GAS for the closed-loop system will also present robustness properties with respect to various persistent external inputs.

The structure of the paper is as follows. In Section II we present all notions used in the paper and formally introduce the problem, whereas Section III contains our main results. Due to space limitations, all proofs are omitted and can be found in [12].

II. Background and definitions

A. Notation

Throughout this paper, we adopt the following notation.

- $\mathbb{R}_+ := [0,\infty)$. Given $x \in \mathbb{R}^n$, we denote by $|x|$ its usual Euclidean norm.
- By $\mathcal{K}$ we denote the set of increasing and continuous functions $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(0) = 0$. A function $\rho \in \mathcal{K}$ is of class $\mathcal{K}_\infty$ if $\lim_{s\to\infty} \rho(s) = \infty$. By $\mathcal{KL}$ we denote the set of functions $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each $t \geq 0$, $\sigma(\cdot,t) \in \mathcal{K}$ and, for each $s \geq 0$, $\sigma(s,\cdot)$ is non-increasing with $\lim_{t\to\infty} \sigma(s,t) = 0$.
- Let $I \subseteq \mathbb{R}$ be a non-empty interval and $\Omega \subset \mathbb{R}^n$ be a non-empty set. By $C^0(I;\Omega)$, we denote the class of continuous functions on I, which take values in Ω. When the interval I is compact, $C^0(I;\Omega)$ is a normed linear space with the norm
$$\|x\|_\infty := \sup_{t\in I}(|x(t)|) = \max_{t\in I}(|x(t)|), \quad \forall x \in C^0(I;\Omega).$$
When $I = [-r,0]$ with $r > 0$ and $a \in (0,1]$ is a constant, we define the Hölder space $C^{0,a}([-r,0]) := \left\{x \in C^0([-r,0];\mathbb{R}^n) : Px < +\infty\right\}$, where
$$Px := \sup_{t,s\in[-r,0],t\neq s}\left(\frac{|x(t)-x(s)|}{|t-s|^a}\right), \tag{1}$$
i.e., the space of Hölder continuous function of exponent $a \in (0,1]$. The Hölder space $\mathcal{X} = C^{0,a}([-r,0])$ with $a \in (0,1]$ is a normed linear space with the norm
$$\|x\|_{\mathcal{X}} := \max\left\{\|x\|_\infty, Px\right\}, \quad \forall x \in C^{0,a}([-r,0]).$$
By virtue of the Arzela-Ascoli theorem, for every $R > 0$, the set $B = \left\{x \in C^{0,a}([-r,0]) : \|x\|_{\mathcal{X}} \leq R\right\}$ is compact in the topology of $C^0([-r,0];\mathbb{R}^n)$.
- Let $I = (a,b)$ be a non-empty open interval. By $L^p(I;\mathbb{R}^n)$ with $p \in [1,+\infty)$, we denote the normed linear space of equivalence classes of Lebesgue measurable functions $x : I \to \mathbb{R}^n$ with $\int_a^b |x(s)|^p\, ds < +\infty$ and, with the norm given by
$$\|x\|_p := \left(\int_a^b |x(s)|^p\, ds\right)^{1/p}, \quad \forall x \in L^p(I;\mathbb{R}^n).$$
By $L^\infty(I;\mathbb{R}^n)$ we denote the normed linear space of equivalence classes of Lebesgue measurable functions $x : I \to \mathbb{R}^n$ with $\sup_{t\in(a,b)} |x(t)| < \infty$ (where $\sup_{t\in(a,b)} |x(t)|$ denotes the essential supremum), with the norm
$$\|x\|_\infty := \sup_{t\in(a,b)} |x(t)|, \quad \forall x \in L^\infty(I;\mathbb{R}^n).$$
- Given $r > 0$, the Sobolev space $W^{1,p}([-r,0])$ for $p \in [1,+\infty]$ denotes the normed linear space of absolutely continuous functions $x : [-r,0] \to \mathbb{R}^n$ with derivative

$\dot{x}$ in $L^p((-r,0);\mathbb{R}^n)$. For the Sobolev space $\mathcal{X} = W^{1,p}([-r,0])$, we use the norm

$$\|x\|_{\mathcal{X}} := \|x\|_\infty + \|\dot{x}\|_p, \quad \forall x \in W^{1,p}([-r,0]),$$

which (by virtue of Theorem 8.8 on pages 212-213 in [2]) is an equivalent norm to the norm $\|x\|_p + \|\dot{x}\|_p$. Notice that $W^{1,\infty}([-r,0]) = C^{0,1}([-r,0])$. By virtue of Hölder's inequality, it follows that $W^{1,p}([-r,0]) \subseteq C^{0,1-1/p}([-r,0])$ for all $p \in (1,+\infty]$. Moreover, by Arzela-Ascoli's theorem and [2, Theorem 8.8, p. 212-213], it follows that, for every $R > 0$, the bounded set

$$B := \left\{ x \in W^{1,p}([-r,0]) : \ \|x\|_\infty + \|\dot{x}\|_p \le R \right\}$$

with $p \in (1,+\infty]$ has a closure $\overline{B}$ in $C^0([-r,0];\mathbb{R}^n)$ which is compact in the topology of $C^0([-r,0];\mathbb{R}^n)$ and satisfies

$$\overline{B} \subseteq \left\{ x \in C^{0,1-\frac{1}{p}}([-r,0] : \ \max\{\|x\|_\infty, Px\} \le R \right\},$$

where Px is defined in (1).

B. Time-delay systems and stability notions

In this work we focus on time-invariant delay systems of the form

$$\dot{x}(t) = f(x_t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $x_t \in C^0([-r,0];\mathbb{R}^n)$ with $r > 0$ being a constant and

$$x_t(s) := x(t+s), \quad \forall s \in [-r,0].$$

The map $f : C^0([-r,0];\mathbb{R}^n) \to \mathbb{R}^n$ with $f(0) = 0$ is assumed to be Lipschitz on bounded sets of $C^0([-r,0];\mathbb{R}^n)$, i.e., there exists a non-decreasing function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $R \ge 0$, the following inequality holds

$$|f(x) - f(y)| \le L(R)\|x-y\|_\infty, \tag{3}$$

for all $x, y \in C^0([-r,0];\mathbb{R}^n)$ with $\|x\|_\infty \le R$, $\|y\|_\infty \le R$.

Given any initial condition $x_0 \in C^0([-r,0];\mathbb{R}^n)$ and any $t \ge 0$, $\phi(t,x_0) \in C^0([-r,0];\mathbb{R}^n)$ denotes x_t, where $x(t) \in \mathbb{R}^n$ is the solution of (2) at time t generated from x_0.

The following properties have been used extensively in the literature of stability for delay systems. In what follows we use the notation C for $C^0([-r,0];\mathbb{R}^n)$.

(LS) *Lyapunov Stability:* For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup\{\|\phi(t,x_0)\|_\infty : \ t \ge 0, x \in C, \|x_0\|_\infty \le \delta\} \le \varepsilon.$$

(GA) *Global Attractivity:* For every $x_0 \in C$,

$$\lim_{t\to\infty} \|\phi(t,x_0)\|_\infty = 0.$$

(UGA) *Uniform Global Attractivity:* For every $\varepsilon, \rho > 0$, there exists $T > 0$ such that

$$\sup\{\|\phi(t,x_0)\|_\infty : t \ge T, x_0 \in C, \|x_0\|_\infty \le \rho\} \le \varepsilon.$$

(LagS) *Lagrange Stability:* For every $\rho > 0$,

$$\sup\{\|\phi(t,x_0)\|_\infty : t \ge 0, x_0 \in C, \|x_0\| \le \rho\} < +\infty.$$

(RFC) *Robust Forward Completeness:* For all $\rho, T > 0$,

$$\sup\Big\{ \|\phi(t,x_0)\|_\infty \ : t \in [0,T], \; x_0 \in C, \|x_0\| \le \rho \Big\} < +\infty.$$

(GAS) *Global Asymptotic Stability:* Both properties LS and GA hold.

(UGAS) *Uniform Global Asymptotic Stability:* There exists $\sigma \in \mathcal{KL}$ such that, for all $x_0 \in C^0([-r,0];\mathbb{R}^n)$,

$$\|\phi(t,x_0)\|_\infty \le \sigma(\|x_0\|_\infty, t), \quad \forall t \ge 0.$$

LS is purely a local property and imposes that solutions remain arbitrarily close to the origin provided that the norm of the initial segment is sufficiently small. GA requires that all solutions converge to the origin. UGA additionally requires that the rate at which solutions converge is uniform on bounded sets of initial states. LagS can be interpreted as solutions' boundedness. RFC requires not only existence of solutions for all forward times, but also that they are bounded over any compact time interval and for initial states in any bounded set. Finally, UGAS employs the classical $\mathcal{KL}$ formalism and readily implies both LS and UGA.

Some of the above properties are related. For example, it is well-known (see Theorem 2.2 on page 62 in [11]) that

$$\text{LS} \wedge \text{UGA} \wedge \text{LagS} \quad \Leftrightarrow \quad \text{UGAS}.$$

Moreover, Lemma 2.1 on page 58 in [11] shows that

$$\text{UGA} \wedge \text{RFC} \quad \Leftrightarrow \quad \text{UGAS}.$$

Some of the above properties are stronger than others; for example, the implications UGA $\Rightarrow$ GA and LagS $\Rightarrow$ RFC hold trivially.

To the best of our knowledge, it is not known whether the implication GAS $\Rightarrow$ UGAS is true or not for delay systems. This implication is true for delay-free (finite-dimensional) systems [24], [29]. Another important property that is valid for delay-free systems is that simple forward completeness (i.e., global existence of solutions for arbitrary initial condition) implies RFC [18], [26]. Whether or not this equivalence holds for time-delay systems is also an open question. This two questions are the main subject of this paper.

III. Main results

A. Uniformity under RFC

The first contribution of this paper is to show that, provided that RFC holds, GAS is equivalent to UGAS.

Theorem 1: For system (2), it holds that

$$\text{GAS} \wedge \text{RFC} \quad \Leftrightarrow \quad \text{UGAS}.$$

In other words, provided that RFC holds, the combination of Lyapunov stability and global attractivity does ensure uniform global asymptotic stability, just like in the finite-dimensional case. It is worth stressing that:

(i) RFC holds automatically for important classes of delay systems such as those ruled by a globally Lipschitz vector field. This reminds the results proved in [17] for globally Lipschitz systems (see Condition 4, p. 128, Definition 28.1, p. 131, Definition 30.2, p. 146, and pages 150, 151 in [17]).

(ii) RFC can often be established by using Lyapunov-like functionals (see [1] for the finite-dimensional case and [11] for the time-delay case). For example, the existence of a functional $U : C^0([-r,0];\mathbb{R}^n) \to \mathbb{R}_+$ which is Lipschitz on bounded sets of $C^0([-r,0];\mathbb{R}^n)$ and for which there exist a function $a \in \mathcal{K}_\infty$ and a constant $\mu \geq 0$ such that the inequalities $U(x) \geq a(|x(0)|)$ and

$$\limsup_{h\to 0^+} \frac{U(P_h(x)) - U(x)}{h} \leq \mu U(x)$$

hold for all $x \in C^0([-r,0];\mathbb{R}^n)$, where $(P_h(x))(s) = x(s+h)$ for $h \geq 0$ and $s \in [-r,-h]$, and $(P_h(x))(s) = x(0) + (s+h)f(x)$ for $h > 0$ and $s \in [-h,0]$, is sufficient to guarantee RFC.

Since GA obviously implies forward completeness, the RFC requirement in Theorem 1 could be removed if one could establish that forward completeness implies RFC for time-delay systems (as it holds in finite dimension). To date, this crucial question remains open.

The fact that LagS trivially ensures RFC, gives us the following corollary. Notice that, differently from [15, Theorem 6.3.1, p. 73], here the LagS property is invoked. Notice also that LS $\wedge$ LagS is equivalent to LS with the additional requirement that the $\delta(\varepsilon)$ for which

$$\sup\left\{\|\phi(t,x_0)\|_\infty : t \geq 0, x \in C, \|x_0\|_\infty \leq \delta(\varepsilon)\right\} \leq \varepsilon$$

can be chosen arbitrarily large for sufficiently large $\varepsilon > 0$.

Corollary 3.1: For system (2), it holds that

$$\text{GAS} \wedge \text{LagS} \quad \Leftrightarrow \quad \text{UGAS}.$$

B. Uniformity in Sobolev and Hölder spaces

The second contribution of this paper is to show that, when working in a Sobolev space $W^{1,p}([-r,0])$ with $p \in (1,+\infty]$ or a Hölder space $C^{0,q}([-r,0])$ with $q \in (0,1]$, the RFC requirement can be removed from the above implications. It is a fact that if the initial condition $x_0 \in C^0([-r,0],\mathbb{R}^n)$ is of class $W^{1,p}([-r,0])$ for some $p \in [1,+\infty]$, then the solution $\phi(t,x_0)$ is of class $W^{1,p}([-r,0])$ for each $t \geq 0$ at which it exists. Similarly, if the initial condition $x_0 \in C^0([-r,0],\mathbb{R}^n)$ is of class $C^{0,q}([-r,0])$ for some $q \in (0,1]$, then the solution $\phi(t,x_0)$ is of class $C^{0,q}([-r,0])$ for each $t \geq 0$ at which it exists. These facts have been utilized in some works on delay systems (see for example [13], [14], [22] with $p = +\infty$ or $q = 1$). Therefore, instead of considering the state space to be $C^0([-r,0];\mathbb{R}^n)$, we may consider as state space the normed linear space $\mathcal{X} = W^{1,p}([-r,0])$ for some $p \in (1,+\infty]$ with

$$\|x\|_{\mathcal{X}} = \|x\|_\infty + \|\dot{x}\|_p, \quad \forall x \in W^{1,p}([-r,0]),$$

or the normed linear space $\mathcal{X} = C^{0,q}([-r,0])$ for some $q \in (0,1]$ with

$$\|x\|_{\mathcal{X}} = \max\left\{\|x\|_\infty,\ Px\right\}, \quad \forall x \in C^{0,q}([-r,0]),$$

where Px was defined in (1).

The change of the state space requires updating of the properties listed above by replacing the sup norm by the Sobolev norm or the Hölder norm. This leads to the following counterparts in which $\mathcal{X}$ may denote either the Sobolev space $W^{1,p}([-r,0])$ for some $p \in (1,+\infty]$ or the Hölder space $C^{0,q}([-r,0])$ for some $q \in (0,1]$.

(LS-$\mathcal{X}$) *Lyapunov Stability:* For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup\left\{\|\phi(t,x_0)\|_{\mathcal{X}} : t \geq 0, x_0 \in \mathcal{X}, \|x\|_{\mathcal{X}} \leq \delta\right\} \leq \varepsilon.$$

(GA-$\mathcal{X}$) *Global Attractivity:* For every $x_0 \in \mathcal{X}$,

$$\lim_{t\to\infty} \|\phi(t,x_0)\|_{\mathcal{X}} = 0.$$

(UGA-$\mathcal{X}$) *Uniform Global Attractivity:* For every $\varepsilon, \rho > 0$, there exists $T > 0$ such that

$$\sup\left\{\|\phi(t,x_0)\|_{\mathcal{X}} : t \geq T, x_0 \in \mathcal{X}, \|x_0\|_{\mathcal{X}} \leq \rho\right\} \leq \varepsilon.$$

(LagS-$\mathcal{X}$) *Lagrange Stability:* For every $\rho > 0$,

$$\sup\left\{\|\phi(t,x_0)\|_{\mathcal{X}} : t \geq 0, x_0 \in \mathcal{X}, \|x_0\|_{\mathcal{X}} \leq \rho\right\} < \infty.$$

(RFC-$\mathcal{X}$) *Robust Forward Completeness:* For every $\rho, T > 0$,

$$\sup\left\{\|\phi(t,x_0)\|_{\mathcal{X}} : t \in [0,T], x_0 \in \mathcal{X}, \|x_0\|_{\mathcal{X}} \leq \rho\right\} < \infty.$$

(GAS-$\mathcal{X}$) *Global Asymptotic Stability:* Both LS-$\mathcal{X}$ and GA-$\mathcal{X}$ hold.

(UGAS-$\mathcal{X}$) *Uniform Global Asymptotic Stability:* There exists $\sigma \in \mathcal{KL}$ such that, for all $x_0 \in \mathcal{X}$,

$$\|\phi(t,x_0)\|_{\mathcal{X}} \leq \sigma(\|x_0\|_{\mathcal{X}}, t), \quad \forall t \geq 0.$$

With these definitions at hand, we are able to show that simple forward completeness (i.e., global existence of solutions for arbitrary initial condition) implies RFC in particular Sobolev or Hölder spaces.

Theorem 2: Let $p \in (1, +\infty]$ be given. Suppose that (2) is forward complete in the sense that for every $x_0 \in C^{0,1-1/p}([-r, 0])$, the solution $x(t)$ of (2) with initial condition x_0 exists for all $t \geq 0$. Then (2) with state space $\mathcal{X} = W^{1,p}([-r, 0])$ or state space $C^{0,1-1/p}([-r, 0])$ owns the RFC-$\mathcal{X}$ property.

Moreover, in this case we are able to show that Lyapunov stability combined with (non-uniform) global attractivity implies uniform global asymptotic stability.

Theorem 3: Let $\mathcal{X}$ denote the Sobolev space $W^{1,p}([-r, 0]$ or the Hölder space $C^{0,1-1/p}([-r, 0])$ for some $p \in (1, +\infty]$. Suppose that (2) is forward complete, in the sense that for every $x_0 \in C^{0,1-1/p}([-r, 0])$, the solution $x(t) \in \mathbb{R}^n$ of (2) with the initial condition x_0 exists for all $t \geq 0$. Then the following equivalence holds for system (2):

$$\text{GAS-}\mathcal{X} \quad \Leftrightarrow \quad \text{UGAS-}\mathcal{X}.$$

Clearly, when $\mathcal{X} = C^{0,1-1/p}([-r, 0])$, the assumption in Theorem 3 that for every $x_0 \in C^{0,1-1/p}([-r, 0])$ the solution $x(t) \in \mathbb{R}^n$ of (2) with the initial condition $x_0 \in C^{0,1-1/p}([-r, 0])$ exists for all $t \geq 0$ is a redundant assumption (since both GAS-$\mathcal{X}$ and UGAS-$\mathcal{X}$ imply this property). Thus, when working with the Hölder spaces $C^{0,1-1/p}([-r, 0])$ with $p \in (1, +\infty]$, the combination of Lyapunov stability and global attractivity is equivalent to uniform global asymptotic stability, just like for finite-dimensional systems.

The stability properties of system (2) viewed in different state spaces are related. The next theorem uses the following stability notion, which provides a $\mathcal{KL}$ bound on the sup norm of the state in terms of the Sobolev norm (when $\mathcal{X} = W^{1,p}([-r, 0])$ or the Hölder norm (when $\mathcal{X} = C^{0,1-1/p}([-r, 0])$) of the initial condition.

(Q-$\mathcal{X}$) There exists $\sigma \in \mathcal{KL}$ such that, for all $x_0 \in \mathcal{X}$,

$$\|\phi(t, x_0)\|_\infty \leq \sigma\left(\|x_0\|_{\mathcal{X}}, t\right), \quad \forall\, t \geq 0.$$

Theorem 4: Let $\mathcal{X}$ denote the Sobolev space $W^{1,p}([-r, 0]$ or the Hölder space $C^{0,1-1/p}([-r, 0])$ for some $p \in (1, +\infty]$. Then the following implications hold for system (2):

$$\text{UGAS} \quad \Rightarrow \quad \text{Q-}\mathcal{X} \quad \Leftrightarrow \quad \text{UGAS-}\mathcal{X}.$$

C. Lyapunov-Krasovskii characterizations of GAS

This section contains our third contribution. Using fundamental properties of delay systems and the converse Lyapunov theory in [11] we obtain the following Lyapunov characterization of the UGAS-$\mathcal{X}$ property when $\mathcal{X}$ denotes either a Sobolev or a Hölder space.

Theorem 5: Let $\mathcal{X}$ denote the space $W^{1,p}([-r, 0]$ or the space $C^{0,1-1/p}([-r, 0])$ for some $p \in (1, +\infty]$. Then the system (2) is UGAS-$\mathcal{X}$ if and only if there exist a functional $V : \mathcal{X} \to \mathbb{R}_+$ which is Lipschitz on bounded sets of $\mathcal{X}$ and functions $a_1, a_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{X}$,

$$a_1(\|x\|_{\mathcal{X}}) \leq V(x) \leq a_2(\|x\|_{\mathcal{X}}) \tag{4}$$

$$V(\phi(t, x)) \leq \exp(-t)V(x), \quad \forall\, t \geq 0. \tag{5}$$

The problem with Theorem 5 is that we were not able to obtain a differential inequality that is equivalent to inequality (5). Indeed, (5) implies the differential inequality

$$\limsup_{t \to 0^+} \frac{V(\phi(t, x)) - V(x)}{t} \leq -V(x), \tag{6}$$

but the differential inequality (6) does not necessarily imply (5) since we do not know whether the mapping $t \mapsto V(\phi(t, x))$ is lower semi-continuous or not (if it were lower semi-continuous then an application of Lemma 2.12 on pages 77-78 in [11] would imply (5)). However, when $p < \infty$, the mapping $t \mapsto V(\phi(t, x))$ is in fact continuous in the topology of $W^{1,p}([-r, 0])$ (see [12, Lemma 1]). Therefore, when $\mathcal{X} = W^{1,p}([-r, 0])$ for some $p \in (1, +\infty)$, we are able to characterize the GAS-$\mathcal{X}$ (hence, the UGAS-$\mathcal{X}$) property through a coercive Lyapunov-Krasovskii functional whose upper right Dini derivative leads to an exponential decay estimate.

Theorem 6: Let $\mathcal{X}$ denote the space $W^{1,p}([-r, 0])$ for some $p \in (1, +\infty)$. Suppose that (2) is forward complete in the sense that for every $x_0 \in C^{0,1-1/p}([-r, 0])$, the solution $x(t) \in \mathbb{R}^n$ of (2) with initial condition x_0 exists for all $t \geq 0$. Then the following statements are equivalent for system (2).

(i) GAS-$\mathcal{X}$

(ii) UGAS-$\mathcal{X}$

(iii) There exist a functional $V : \mathcal{X} \to \mathbb{R}_+$ which is Lipschitz on bounded sets of $\mathcal{X}$ and functions $a_1, a_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{X}$,

$$a_1(\|x\|_{\mathcal{X}}) \leq V(x) \leq a_2(\|x\|_{\mathcal{X}}) \tag{7}$$

$$\limsup_{t \to 0^+} \frac{V(\phi(t, x)) - V(x)}{t} \leq -V(x) \tag{8}$$

(iv) There exist a functional $V : \mathcal{X} \to \mathbb{R}_+$ which is Lipschitz on bounded sets of $\mathcal{X}$, a continuous positive definite function $Q : \mathbb{R}^n \to \mathbb{R}_+$ and functions $a_1, a_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{X}$,

$$a_1(|x(0)|) \leq V(x) \leq a_2(\|x\|_{\mathcal{X}}); \tag{9}$$

$$\limsup_{t \to 0^+} \frac{V(\phi(t, x)) - V(x)}{t} \leq -Q(x(0)). \tag{10}$$

It can be easily checked that (7)-(8) imply (9)-(10). Indeed, (7)-(8) are conditions for a coercive Lyapunov-Krasovskii functional with dissipation rate involving the whole functional itself, whereas the functional in (9)-(10) is not required to be coercive and its dissipation rate merely involves the current value of the solution. As discussed in [4], [5], [3], the use of non-coercive Lyapunov-Krasovskii functional with point-wise dissipation rate is often more convenient to ensure GAS in practice.

The assumption that (2) is forward complete in the sense that for every $x_0 \in C^{0,1-1/p}([-r, 0])$, the solution $x(t) \in \mathbb{R}^n$ of (2) with initial condition x_0 exists for all $t \geq 0$ is not redundant in Theorem 6. This assumption is needed only for the implication (i) $\Rightarrow$ (ii) (guaranteed by means of Theorem 3). Indeed, the property GAS-$\mathcal{X}$ does not guarantee the required forward completeness assumption when $\mathcal{X} = W^{1,p}([-r, 0])$.

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