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Abstract
In this paper it is proposed to extend the result described in Khalil and Praly (2014) and the references therein, regarding the high-gain observer-based nonlinear control to the case of systems with diffusion sensor dynamic. Based on some usual hypotheses, we provide sufficient conditions involving the high-gain parameter and the length on the PDE sensor. In fact it is brought into light an explicit trade off between them: the larger the observer gain, the smaller the length of the PDE sensor needs to be. The stability analysis of the closed loop system is based on a Lyapunov functional.

Keywords: Output feedback, High-gain observers, Cascade ODE-PDE systems.

1. INTRODUCTION

The present paper deals with the design of output feedback control for a class of nonlinear cascade ODE-PDE systems. Throughout the past decades, the high-gain observers have been used extensively for the design of output feedback control of nonlinear systems, see Khalil and Praly (2014) and the references therein. An important advantage of using the high-gain observers is that they can recover the performances of state feedback control in the sense that, for instance, the trajectories of the system under output feedback approach those under state feedback as the observer gain increases.

Moreover, from the work Atassi and Khalil (1999), it is well known that the separation principle holds not only because the observer gain is made high, but also because, by designing the feedback control as a globally bounded function, the state of the plant is protected from the peaking phenomenon Khalil (1996) when the high-gain observer estimates are used instead of the true states.

In fact, the design of output feedback control for cascade ODE-PDE systems has not been yet very much studied, one can cite for instance the work Krstic (2009) for a linear ODE or the work Wu (2013) where a nonlinear ODE is considered. However in this later work, the nonlinear ODE is restrictive and the method, based on LMIs, does not provide explicit conditions regarding the length that the PDE must satisfy in order to ensure global exponential stability of the overall system.

In the present paper we propose to extend the design of high-gain observer-based output feedback control for nonlinear systems with sensors described by heat PDEs. More precisely, we will derive explicit sufficient conditions, involving both the high-gain and the length of the PDE, ensuring exponential convergence of the overall closed cascade ODE-PDE. It has also to be noticed that the observer designed here is more simple than those designed in Ahmed-Ali et al. (2015) and Ahmed-Ali et al. (2019a) for the same cascade ODE-PDE systems which used backstepping technics. A sampled-data version of our algorithm is also presented with its convergence conditions. A conference version of the present work has been presented at Ahmed-Ali et al. (2020). The main difference between the two versions is in the the fact that this version contains supplementary results by treating the sampled-data case.

Notations and preliminaries

Throughout the paper the superscript T stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|.|$, $\mathbf{I}_n$ is the $n \times n$ identity matrix, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$, means that P is symmetric and positive definite. In matrices, symmetric terms are denoted $*$; $\lambda_{min}(P)$ $(resp.\lambda_{max}(P))$ denotes the smallest (resp.

largest) eigenvalue. $L_2(0,D)$ is the Hilbert space of square integrable functions $z :\to \mathbf{R}$ with the corresponding norm $\|z\|_{L_2} = \sqrt{\int_0^D z^2(x)dx}$. $\mathcal{H}^k(0,D)$ is the Sobolev space of functions $z : [0,D] \to \mathbf{R}$ having k square integrable weak derivatives. Given a two-argument function $u(x,t)$, then its partial derivatives are denoted $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$. $u[t]$ and $u_x[t]$ refer to the functions defined on $0 \leq x \leq D$ by $(u[t])(x) = u(x,t)$ and $(u_x[t])(x) = \frac{\partial u(x,t)}{\partial x}$. In this paper the Halany's type Inequalities, Fridman and Blighovsky (2012) summarized in the following lemma will be used.

Lemma 1. Let $0 < \delta_1 < 2\delta$ and let $V : [t_0 - h, \infty) \to [0, \infty)$ be an absolutely continuous function which satisfies

$$\dot{V}(t) \leq -2\delta V(t) + \delta_1 sup_{-h \leq s \leq 0} V(t+s)$$

Then

$$V(t) \leq e^{-2\alpha(t-t_0)} sup_{-h \leq s \leq 0} V(s)$$

where α is the unique positive solution of equation

$$\alpha = \delta - \frac{\delta_1 e^{2\alpha h}}{2}$$

2. PROBLEM STATEMENT

Let us consider the following class of systems in the state space $\mathbb{R}^n \times L_2(0,D)$.

$$\begin{cases} \dot{X}_i = X_{i+1}, \quad & i = 1, \dots, n-1 \\ \dot{X}_n = f(X, v) \\ u(D,t) = X_1 \\ u_x(0,t) = 0 \\ u_t = u_{xx}, \quad x \in [0,D] \\ y(t) = u(0,t) \end{cases} \tag{1}$$

where v and y represent respectively the input and the output of the above system. Throughout the paper, we assume the following hypotheses:

H1: The function f is continuous globally Lipschitz in both X and v with a Lipschitz constant K_0.

H2: : There exists a continuous globally Lipschitz function $\alpha(X)$, such that the following dynamical system

$$\begin{cases} \dot{X}_i = X_{i+1} \\ \dot{X}_n = f(X, \alpha(X)) \end{cases}$$

is globally exponentially stable.

Using the converse Lyapunov Theorem Khalil (1996) we can say that there exists a function $V_0(X) > 0$ and positive parameters $c_i, i = 1, \dots, 4$ such that:

$$\begin{cases} c_1|X|^2 \leq V_0(X) \leq c_2|X|^2 \\ |\frac{\partial V_0}{\partial X_i}| \leq c_3|X| \\ \sum_{i=1}^{n-1} \frac{\partial V_0}{\partial X_i} X_{i+1} + \frac{\partial V_0}{\partial X_n} f(X, \alpha(X)) \leq -c_4|X|^2 \end{cases} \tag{2}$$

for all X.

If we consider the changes of coordinates $p(x,t) = u(x,t) - X_1$ then the above system can be written as follows

$$\begin{cases} \dot{X}_i = X_{i+1}, \quad & i = 1, \dots, n-1 \\ \dot{X}_n = f(X, v) \\ p(D,t) = 0 \\ p_x(0,t) = 0 \\ p_t = p_{xx} - X_2, \quad x \in [0,D] \\ y(t) = p(0,t) + X_1 \end{cases} \tag{3}$$

3. OUTPUT FEEDBACK DESIGN

Based on the above hypotheses, we propose the following high-gain observer-based output feedback control:

$$\begin{cases} \dot{Z}_i = Z_{i+1} - l_i\theta^i(\hat{u}(0,t) - y), \quad & i = 1, \dots, n-1 \\ \dot{Z}_n = f(Z, v) - l_n\theta^n(\hat{u}(0,t) - y) \\ v = \alpha(Z) \\ \hat{u}(D,t) = Z_1 \\ \hat{u}_x(0,t) = 0 \\ \hat{u}_t = \hat{u}_{xx} - l_1\theta(\hat{u}(0,t) - y) \quad x \in [0,D] \end{cases} \tag{4}$$

The vector gains $L = (l_1, \cdots, l_n)^\top \in \mathbb{R}^{n\times 1}$ is chosen such that the matrix $(A - LC)$ is Hurwitz where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & 0 & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times 2}$$

and $C = [1, 0, \cdots, 0] \in \mathbb{R}^{1\times n}$. Let us use the change of coordinates

$$\hat{p}(x,t) = \hat{u}(x,t) - Z_1$$

then the observer (4) can be expressed as follows

$$\begin{cases} \dot{Z}_i = Z_{i+1} - l_i\theta^i(\hat{u}(0,t) - y), \quad & i = 1, \dots, n-1 \\ \dot{Z}_n = f(Z, v) - l_n\theta^n(\hat{u}(0,t) - y) \\ v = \alpha(Z) \\ \hat{p}(D,t) = 0 \\ \hat{p}_x(0,t) = 0 \\ \hat{p}_t = \hat{p}_{xx} - Z_2 \quad x \in [0,D] \end{cases} \tag{5}$$

Remark 2. The existence and uniqueness of solutions of system (5) can be analyzed by using Theorem 2.1 of Karafyllis and Krstic (2017). As in Remark 4 of Ahmed-Ali et al. (2019b) where the considered system is very similar to the one considered here, we can say that for every $\hat{p}[0] \in \mathcal{C}^2(0,D)$ with $(\hat{p}[0])(D) = (\hat{p}[0])'(0) = 0$, there exists an unique solution $\hat{p} \in ([0,D] \times \mathcal{C}^0[0,\infty)) \cap ([0,D] \times \mathcal{C}^1(0,\infty))$ with $\hat{p}[t] \in \mathcal{C}^2(0,D)$ for all $t \geq 0$. We can also use the same arguments for system (3) to say that for every $p[0] \in \mathcal{C}^2(0,D)$ with $(p[0])(D) = (p[0])'(0) = 0$, there exists an unique solution $p \in ([0,D] \times \mathcal{C}^0[0,\infty)) \cap ([0,D] \times \mathcal{C}^1(0,\infty))$ with $p[t] \in \mathcal{C}^2(0,D)$ for all $t \geq 0$ and $\eta \in (\mathcal{C}^1(0,\infty); \mathbb{R}^n \times \mathbb{R}^n)$. From this we can also claim that both $u \in ([0,D] \times \mathcal{C}^0[0,\infty)) \cap ([0,D] \times \mathcal{C}^1(0,\infty))$, $u[t] \in \mathcal{C}^2(0,D)$, $\hat{u} \in ([0,D] \times \mathcal{C}^0[0,\infty)) \cap ([0,D] \times \mathcal{C}^1(0,\infty))$ and $\hat{u}[t] \in \mathcal{C}^2(0,D)$.

Let us consider the dynamical error system $e = Z - X$ and $\tilde{u} = \hat{u} - u$, then for any $x \in [0,D]$, we obtain

$$\begin{cases} \dot{e}_i = e_{i+1} - l_i\theta^i\tilde{u}(0,t), \qquad i=1,\dots,n-1 \\ \dot{e}_n = f(Z,\alpha(Z)) - f(X,\alpha(Z)) - l_n\theta^n\tilde{u}(0,t) \\ \dot{X}_i = X_{i+1}, \qquad i=1,\dots,n-1 \\ \dot{X}_n = f(X,\alpha(Z)) \\ \tilde{u}(D,t) = e_1 \\ \tilde{u}_x(0,t) = 0 \\ \tilde{u}_t = \tilde{u}_{xx} - l_1\theta\tilde{u}(0,t), \quad x\in[0,D] \end{cases} \tag{6}$$

Now if we introduce the change of coordinate

$$w(x,t) = \tilde{u}(x,t) - e_1,$$

then we have

$$\begin{cases} \dot{e}_i = e_{i+1} - l_i\theta^i e_1 - l_i\theta^i w(0,t) \qquad i=1,\dots,n-1 \\ \dot{e}_n = f(Z,\alpha(Z)) - f(X,\alpha(Z)) - l_n\theta^n e_1 - l_n\theta^n w(0,t) \\ \dot{X}_i = X_{i+1}, \qquad i=1,\dots,n-1 \\ \dot{X}_n = f(X,\alpha(Z)) \\ w(D,t) = 0 \\ w_x(0,t) = 0. \\ w_t = w_{xx} - e_2,, \quad x\in[0,D] \end{cases} \tag{7}$$

By using the classical change of coordinates $\xi_i = \theta^{1-i}e_i$, we derive

$$\begin{cases} \dot{\xi}_i = \theta\xi_{i+1} - \theta l_i\xi_1 - l_i\theta w(0,t), \qquad i=1,\dots,n-1 \\ \dot{\xi}_n = \theta^{1-n}\left[f(Z,\alpha(Z)) - f(X,\alpha(Z))\right] - \theta l_n\xi_1 - \theta l_n w(0,t) \\ \dot{X}_i = X_{i+1}, \qquad i=1,\dots,n-1 \\ \dot{X}_n = f(X,\alpha(Z)) \\ `w(D,t) = 0 \\ w_x(0,t) = 0 \\ w_t = w_{xx} - \theta\xi_2 \end{cases}$$

Let us now introduce the following augmented vector state

$$\eta = [\xi, X]^T, \quad \text{where} \quad \xi = (\xi_1,\dots,\xi_n)^T.$$

Then the above system can be written as

$$\begin{cases} \dot{\eta} = F_0(\eta, w(0,t)) \\ w(D,t) = 0 \\ w_x(0,t) = 0 \\ w_t = w_{xx} - \theta\xi_2 \end{cases} \tag{8}$$

where $F_0(\eta, w(0,t))$ is given by

$$\begin{pmatrix} \theta\xi_{i+1} - \theta l_i\xi_1 - l_i\theta w(0,t), \qquad i=1,\dots,n-1 \\ \theta^{1-n}\left[f(\Delta\xi + X,\alpha(Z)) - f(X,\alpha(Z))\right] - \theta l_n\xi_1 - \theta l_n w(0,t) \\ X_{i+1}, \qquad i=1,\dots,n-1 \\ f(X,\alpha(\Delta\xi + X)) \end{pmatrix}.$$

where $\Delta = diag(1,\dots,\theta^{n-1})$. Notice that since $w(D,t) = 0$, then $w(0,t) = -\int_0^D w_x(x,t)dx$.

Theorem 3. Consider the system (1). Then for all $(X(0),Z(0)) \in (\mathbb{R}^n\times\mathbb{R}^n)$ and $(u[0],\hat{u}[0]) \in (\mathcal{C}^2(0,D)\times\mathcal{C}^2(0,D))$ with $(u[0])(D) = X_1(0), (u[0])'(0) = 0$, $(\hat{u}[0])(D) = Z_1(0), (\hat{u}[0])'(0) = 0$, there exist θ_0 and $D^\star(\theta)$ such that $\forall\theta > \theta_0$ and $\forall D \in (0, D^\star(\theta))$ the following inequality holds:

$$|X| + |e| + \int_0^D \tilde{u}^2(x,t)dx \le M_1 e^{-\sigma_1 t}$$

for some positive constant $M_1 > 0$ and $\sigma_1 > 0$ where M_1 depends on the initial conditions.

Proof. In order to prove the exponential stability of the system (8), we will divide the proof in three parts: for the infinite dimensional sub-system, for the finite dimensional one and finally for the overall error system.

Infinite dimensional sub-system

In this first part we will analyse the sub-system

$$\begin{cases} w_t = w_{xx} - \theta\xi_2 \\ w(D,t) = 0 \\ w_x(0,t) = 0 \end{cases} \tag{9}$$

In order to do this, we consider the following functional :

$$W = \frac{1}{2}\int_0^D w^2(x,t)dx + \frac{1}{2}\int_0^D w_x^2(x,t)dx$$

Then, the time derivative of W(t) along the trajectory of the system (5) is

$$\begin{aligned} \dot{W}(t) &= \int_0^D w(x,t)w_t(x,t)dx + \int_0^D w_x(x,t)w_{xt}(x,t)dx \\ &= \int_0^D w(x,t)(w_{xx}(x,t) - \theta\xi_2)dx \\ &\quad + \int_0^D w_x(x,t)w_{tx}(x,t)dx \end{aligned}$$

From the fact that $w(D,t) = 0$, then we have $w_t(D,t) = 0$. From this and by using the integration by parts on $[0,D]$, we can easily derive that

$$\begin{aligned} \dot{W} = &-\int_0^D w_x^2(x,t)dx - \theta\int_0^D w(x,t)\xi_2 dx - \int_0^D w_{xx}^2(x,t)dx \\ &+\theta\int_0^D w_{xx}\xi_2 dx \end{aligned}$$

Using Young's inequality leads to

$$\begin{aligned} \dot{W} \le &-\int_0^D w_x^2(x,t)dx - \frac{1}{2}\int_0^D w_{xx}^2(x,t)dx \\ &+\frac{1}{2}\int_0^D w^2(x,t)dx + D\theta^2|\xi_2|^2 \end{aligned}$$

Using Wirtinger's inequality Fridman and Blighovsky (2012), we have

$$\begin{aligned} \dot{W} \le &-(\frac{\pi^2}{4D^2} - \frac{1}{2})\int_0^D w^2(x,t)dx \\ &-\frac{\pi^2}{8D^2}\int_0^D w_x^2(x,t)dx + D\theta^2|\xi_2|^2. \end{aligned}$$

Since $|\xi_2| \le |\eta|$, then

$$\begin{aligned} \dot{W} \le &-(\frac{\pi^2}{4D^2} - \frac{1}{2})\int_0^D w^2(x,t)dx \\ &-\frac{\pi^2}{8D^2}\int_0^D w_x^2(x,t)dx + D\theta^2|\eta|^2 \end{aligned}$$

Finite-dimensional system

Let us now analyse the unperturbed sub-system
$$\dot{\eta} = F_0(\eta, 0).$$

By considering the following Lyapunov function,

$$V(\eta) = \frac{1}{\theta^{2(n-1)}} V_0(X) + \xi^T P \xi$$

where P is a positive definite symmetric matrix which satisfies
$$P(A - LC) + (A - LC)^T P = -\mathbf{I}_n$$

then, we can derive the following inequalities satisfied by V:

$$\begin{cases} c_1' |\eta|^2 \leq V(\eta) \leq c_2' |\eta|^2 \\ |\frac{\partial V}{\partial \eta}| \leq c_3' |\eta| \end{cases}$$

where
$$\begin{cases} c_1' = \min\left\{\frac{c_1}{\theta^{2(n-1)}}, \lambda_{min}(P)\right\} \\ c_2' = \max\left\{\frac{c_2}{\theta^{2(n-1)}}, \lambda_{max}(P)\right\} \\ c_3' = \max\left\{\frac{c_3}{\theta^{2(n-1)}}, 2\lambda_{max}(P)\right\} \end{cases}$$

with c_1, c_2 and c_3 defined in (2).

Now let us compute the derivative of V along the solution of the unperturbed system, we obtain,

$$\dot{V} = \frac{\partial V}{\partial \eta} F_0(\eta, 0).$$

After some computations as in Khalil and Praly (2014), we can easily derive the following inequality:

$$\dot{V} \leq -\frac{c_4}{2\theta^{2(n-1)}} |X|^2 - \left[\theta - 2\lambda_{max}(P) K_0 - \frac{2c_3^2}{c_4} K_0^2\right] |\xi|^2.$$

for some positive constant K_0 independent of θ, and c_4 defined in (2).

Choosing $\theta > \theta_0$ such that
$$\theta_0 = \max\left\{2\lambda_{max}(P) K_0 + \frac{2c_3^2}{c_4} K_0^2 + \frac{c_4}{2}, 1\right\} \qquad (10)$$

we deduce that
$$\dot{V} \leq -c_4' |\eta|^2$$

with
$$c_4' = \frac{c_4}{2\theta^{2(n-1)}}.$$

From this, we deduce that the unperturbed system is globally uniformly exponentially stable.

Now let us compute the derivative $\dot{V}$ along the sub-system $\dot{\eta} = F_0(\eta, w(0, t))$, we obtain,

$$\dot{V} = \frac{\partial V}{\partial \eta} F_0(\eta, w(0, t))$$

which can be rewritten as follows:

$$\dot{V} = \frac{\partial V}{\partial \eta} F_0(\eta, 0) + \frac{\partial V}{\partial \eta} \left(F_0(\eta, w(0, t)) - F_0(\eta, 0)\right)$$

Using the above inequalities, we easily derive that

$$\dot{V} \leq -c_4' |\eta|^2 + |\frac{\partial V}{\partial \eta}| \left| \left(F_0(\eta, w(0, t)) - F_0(\eta, 0)\right) \right|$$

Using again Young's inequality, this leads to:

$$\dot{V} \leq -c_4' |\eta|^2 + \beta |\frac{\partial V}{\partial \eta}|^2 + \frac{1}{\beta} \left| \left(F_0(\eta, w(0, t)) - F_0(\eta, 0)\right) \right|^2$$

for $\beta > 0$.

On the other hand, we can easily see that, from the definition of F_0,

$$\left| \left(F_0(\eta, w(0, t)) - F_0(\eta, 0)\right) \right| \leq \theta |L| |w(0, t)|$$

which also gives
$$\left| \left(F_0(\eta, w(0, t)) - F_0(\eta, 0)\right) \right| \leq \theta |L| \int_0^D |w_x(x, t)| dx$$

Now using the inequality of Cauchy-Schwarz, we obtain

$$\left| \left(F_0(\eta, w(0, t)) - F_0(\eta, 0)\right) \right|^2 \leq \theta^2 |L|^2 D \int_0^D w_x^2(x, t) dx$$

and from this we derive that

$$\dot{V} \leq -(c_4' - \beta c_3'^2) |\eta|^2 + \frac{1}{\beta} |\theta^2 |L|^2 D \int_0^D w_x^2(x, t) dx$$

Stability of the overall error system

At this stage, we now consider the Lyapunov functionnal $W_1 = W + V$. Then from the two previous parts, its time derivative satisfies the following inequality

$$\dot{W}_1 \leq -\left[c_4' - \beta {c_3'}^2 - D\theta^2\right] |\eta|^2 - \left[\frac{\pi^2}{4D^2} - \frac{1}{2}\right] \int_0^D w^2(x, t) dx - \left[\frac{\pi^2}{8D^2} - \frac{1}{\beta} \theta^2 |L|^2 D\right] \int_0^D w_x^2(x, t) dx \qquad (11)$$

For any $\epsilon > 0$, we can also write :

$$\dot{W}_1 + 2\epsilon W_1 \leq -\left[c_4' - \beta {c_3'}^2 - D\theta^2 - 2\epsilon c_2'\right] |\eta|^2 - \left[\frac{\pi^2}{4D^2} - \frac{1}{2} - \epsilon\right] \int_0^D w^2(x, t) dx - \left[\frac{\pi^2}{8D^2} - \frac{1}{\beta} \theta^2 |L|^2 D - \epsilon\right] \int_0^D w_x^2(x, t) dx \qquad (12)$$

In order to ensure the exponential stability, it is sufficient to set $\epsilon \to 0$, and it is not difficult to see that we need to choose β sufficiently small so that $c_4' - \beta {c_3'}^2 > 0$ and we have to choose D such that

$$D < D^\star(\theta) = \min\left\{ \frac{1}{\theta^2}(c_4' - \beta {c_3'}^2), \frac{\pi}{\sqrt{2}}, \left(\frac{\pi^2\beta}{8\theta^2|L|^2}\right)^{1/3} \right\} \tag{13}$$

This ends the proof.

Remark: The expression for $D^*(\theta)$ shows that as θ increases, $D^*(\theta)$ decreases. Hence there is a trade off between θ and D; the larger the observer gain θ, the smaller the length of the PDE sensor.

4. EXTENSION TO SAMPLED-DATA CASE

In this section, we propose an extension of the above result to the case where the output is available only at sampling instants t_k:

$$0 = t_0 < t_1 < \ldots, < t_k < \ldots, \lim_{i\to\infty} t_k = \infty$$

with $t_{k+1} - t_k \leq h$. For this case, we propose the following high-gain observer-based output feedback control:
for $t \in [t_k, t_{k+1})$,

$$\begin{cases} \dot{Z}_i = Z_{i+1} - l_i\theta^i(\hat{u}(0,t_k) - y_{t_k}), \qquad i = 1,\ldots,n-1 \\ \dot{Z}_n = f(Z,v) - l_n\theta^n(\hat{u}(0,t_k) - y_{t_k}) \\ v = \alpha(Z) \\ \hat{u}(D,t) = Z_1 \\ \hat{u}_x(0,t) = 0 \\ \hat{u}_t = \hat{u}_{xx} - l_1\theta(\hat{u}(0,t_k) - y_{t_k}) \quad x \in [0,D] \end{cases} \tag{14}$$

Using the same computations and change of coordinates as the continuous time case, we easily derive the following error system :
for $t \in [t_k, t_{k+1})$

$$\begin{cases} \dot{\xi}_i = \theta\xi_{i+1} - \theta l_i \xi_1(t_k) - l_i\theta w(0,t_k), \qquad i = 1,\ldots,n-1 \\ \dot{\xi}_n = \theta^{1-n}\left[f(Z,\alpha(Z)) - f(X,\alpha(Z))\right] \\ -\theta l_n\xi_1(t_k) - \theta l_n w(0,t_k) \\ \dot{X}_i = X_{i+1}, \qquad i = 1,\ldots,n-1 \\ \dot{X}_n = f(X,\alpha(Z)) \end{cases}$$

and

$$\begin{cases} w_t = w_{xx} - \theta\xi_2 \\ w(D,t) = 0 \\ w_x(0,t) = 0 \end{cases}$$

Remark 4. The analysis of well posedness done in the above section (continuous-time case) remains valid for each interval $[t_k, t_{k+1})$. This allows us to also ensure the existence of solutions with the same properties than the continuous-time case.

Since $w(D, t_k) = 0$ then

$$w(0,t_k) = -\int_0^D w_x(x,t_k)dx$$

Then the above system can be obviously re-written as follows :

$$\begin{aligned} &\dot{\xi} = \theta(A - LC)\xi - \theta LC(\xi(t_k) - \xi(t)) \\ &+b\theta^{1-n}\left[f(\Delta\xi + X, \alpha(Z)) - f(X,\alpha(Z))\right] + \theta L\textstyle\int_0^D w_x(x,t_k)dx, \\ &\dot{X}_i = X_{i+1}, \qquad i = 1,\ldots,n-1 \\ &\dot{X}_n = f(X,\alpha(Z)) \\ &w_t = w_{xx} - \theta\xi_2 \\ &w(D,t) = 0 \\ &w_x(0,t) = 0 \end{aligned} \tag{15}$$

Where $b^T = \underbrace{(0,0,\ldots,0,1)}_{n}$. Now, in order to study the convergence of the above error system, we consider the following Lyapunov functional inspired from the work of Selivanov and Fridman (2016) :

$$\begin{aligned} W_2 &= W_1 + W_3 \\ W_3 &= Wh^2e^{2\delta h}\int_{t_i}^t e^{2\delta(s-t)}|\dot{\xi}(s)|^2 ds - \frac{\pi^2}{4}W\int_{t_i}^t |\xi(s) - \xi(t_i)|^2 ds \end{aligned} \tag{16}$$

where δ and W are two positive constants which will be defined latter. By using generalized Wirtinger's inequality, we easily deduce that both W_3 is nonnegative and does not grow at the jumps occurring at instants t_i Selivanov and Fridman (2016). Moreover we have

$$\dot{W}_3 + 2\delta W_3 \leq Wh^2e^{2\delta h}|\dot{\xi}(t)|^2 - \frac{\pi^2}{4}W|\xi(t) - \xi(t_i)|^2 \tag{17}$$

Using Young's inequality, we have

$$\begin{aligned} |\dot{\xi}(t)|^2 &\leq 4\theta^2\left[|A - LC| + K_0\right]^2|\xi(t)|^2 \\ &+ 4\theta^2|L|^2|\xi(t) - \xi(t_k)|^2 + 4\theta^2|L|^2 D\int_0^D w_x^2(x,t_k)dx \end{aligned}$$

then

$$\begin{aligned} \dot{W}_3 + 2\delta W_3 &\leq Wh^2e^{2\delta h}\left[|A - LC| + K_0\right]^2|\xi(t)|^2 \\ &+ Wh^2e^{2\delta h}4\theta^2|L|^2 D\int_0^D w_x^2(x,t_k)dx \\ &- \left[\frac{\pi^2}{4}W - Wh^2e^{2\delta h}4\theta^2|L|^2\right]|\xi(t) - \xi(t_i)|^2 \end{aligned}$$

Now, let us compute the time derivative of W_2 along systems (15), then after some simple computations, we deduce the following inequality

$$\begin{aligned} \dot{W}_2 \leq &- \left[c_4' - \delta {c_3'}^2 - D\theta^2\right]|\eta|^2 - \left[\frac{\pi^2}{4D^2} - \frac{1}{2}\right]\int_0^D w^2(x,t)dx \\ &- \frac{\pi^2}{8D^2}\int_0^D w_x^2(x,t)dx + \frac{2}{\delta}\theta^2|L|^2 D\int_0^D w_x^2(x,t_k)dx - 2\delta W_3 \\ &+ Wh^2e^{2\delta h}\left[|A - LC| + K_0\right]^2|\xi(t)|^2 \\ &+ Wh^2e^{2\delta h}4\theta^2|L|^2 D\int_0^D w_x^2(x,t_k)dx \\ &- \left[\frac{\pi^2}{4}W - Wh^2e^{2\delta h}4\theta^2|L|^2 - \frac{2}{\delta}\theta^2|L|^2\right]|\xi(t) - \xi(t_i)|^2 \end{aligned} \tag{18}$$

which also gives

$$\begin{aligned}
&\dot{W}_2 + 2\delta W_2 \leq \\
&\quad - \left[c_4' - \delta c_3'^2 - D\theta^2 - Wh^2 e^{2\delta h}\left[|A - LC| + K_0\right]^2 - 2\delta c_2'\right]|\eta|^2 \\
&\quad - \left[\frac{\pi^2}{4D^2} - \frac{1}{2} - \delta\right]\int_0^D w^2(x,t)dx \\
&\quad - \left[\frac{\pi^2}{8D^2} - \delta\right]\int_0^D w_x^2(x,t)dx \\
&\quad + \left[\frac{2}{\delta}\theta^2|L|^2 D + Wh^2 e^{2\delta h}4\theta^2|L|^2 D\right]\int_0^D w_x^2(x,t_k)dx \\
&\quad - \left[\frac{\pi^2}{4}W - Wh^2e^{2\delta h}4\theta^2|L|^2 - \frac{2}{\delta}\theta^2|L|^2\right]|\xi(t) - \xi(t_i)|^2.
\end{aligned} \tag{19}$$

Now let us set δ sufficiently small such that $c_4' - \delta c_3'^2 > 0$. For doing this, we can choose $\delta = \frac{1}{\theta^{2n}}$. If we set $W = \frac{4}{\pi^2}\left[\delta|L|^2 + \frac{2}{\delta}\theta^2|L|^2\right]$ and if the sampling period h satisfies $Wh^2e^{2\delta h}4\theta^2 \leq \delta$, we ensure that

$$\frac{\pi^2}{4}W - Wh^2e^{2\delta h}4\theta^2|L|^2 - \frac{2}{\delta}\theta^2|L|^2 > 0$$

and consequently we have

$$\begin{aligned}
\dot{W}_2 + 2\delta W_2 \leq & - \left[c_4' - \delta c_3'^2 - D\theta^2 - \delta\left[|A - LC| + K_0\right]^2 - 2\delta c_2'\right]|\eta|^2 \\
& - \left[\frac{\pi^2}{4D^2} - \frac{1}{2} - \delta\right]\int_0^D w^2(x,t)dx \\
& - \left[\frac{\pi^2}{8D^2} - \delta\right]\int_0^D w_x^2(x,t)dx \\
& + \left[\frac{2}{\delta}\theta^2|L|^2 D + Wh^2e^{2\delta h}4\theta^2|L|^2 D\right]\int_0^D w_x^2(x,t_k)dx
\end{aligned} \tag{20}$$

At this stage, suppose that $D < \min\left(\frac{\delta}{\theta^2}, \frac{\pi}{2\sqrt{2\delta}}, \frac{\pi}{2}\sqrt{\frac{1}{\delta+\frac{1}{2}}}\right)$, and if choose θ sufficiently high so that

$$2\delta c_2' < \left[c_4' - \delta c_3'^2 - D\theta^2 - \delta\left[|A - LC| + K_0\right]^2\right]$$

which is equivalent to

$$\left[2c_2' + \left(1 + \left[|A - LC| + K_0\right]^2 + c_3'^2\right)\right]\frac{1}{\theta^{2n}} \leq c_4' \tag{21}$$

we conclude that

$$\dot{W}_2 \leq -2\delta W_2 + 2\left[\frac{2}{\delta}\theta^2|L|^2 D + \delta|L|^2 D\right]W_2(t_k) \tag{22}$$

Using the fact that $t_k = t - \tau(t)$ where $\tau(t) = t - t_k$, $\forall t \in [t_k, t_{k+1})$, $\tau(t) \leq h$ and by applying Lemma 1, we can say that the system (15) is exponentially convergent if

$$\frac{2}{\delta}\theta^2|L|^2 D + \delta|L|^2 D < \delta. \tag{23}$$

Combining this, with the above conditions, we deduce that for all θ, satisfying conditions (10), (21), and if

$$\begin{cases}
\delta = \frac{1}{\theta^{2n}}, W = \frac{4}{\pi^2}\left[\delta|L|^2 + \frac{2}{\delta}\theta^2|L|^2\right] \\
D < D^*(\theta) = \min\left(\frac{\delta}{\frac{2}{\delta}|L|^2\theta^2 + \delta|L|^2}, \frac{\delta}{\theta^2}, \frac{\pi}{2\sqrt{2\delta}}, \frac{\pi}{2}\sqrt{\frac{1}{\delta+\frac{1}{2}}}\right) \\
Wh^2e^{2\delta h}4\theta^2 \leq \delta
\end{cases} \tag{24}$$

then system (15) is exponentially convergent. It has to be noticed that since the function $Wh^2e^{2\delta h}4\theta^2$ is an increasing function then, $\forall\theta$, there exists $\bar{h}(\theta)$ such that $\forall h \in (0, \bar{h}(\theta))$ the inequality $Wh^2e^{2\delta h}4\theta^2 \leq \delta$ holds. From this we can state the following Theorem :

Theorem 5. Consider system (1). Then, there exist θ_1, $D^\star(\theta)$, $\bar{h}(\theta)$ such that $\forall\theta > \theta_1$, $\forall D \in (0, D^\star(\theta))$ and $\forall h \in (0, \bar{h}(\theta))$, the observation error system (15) converges exponentially to zero.

5. CONCLUSION

An extension of the design of high-gain observer-based output feedback control for nonlinear systems with sensors described by heat PDEs has been proposed by bringing into light explicitly the expected trade-off between the gain of the observer and the length of the PDE. Further works will be undertaken in the same vein by i) considering actuators PDEs and ii) by relaxing the global exponential assumption. This later extension will require careful analysis of the peaking phenomenon. On the other hand it's well known that a heat equation can be decoupled into a finite dimensional unstable part and an infinite dimensional stable one. We are currently investigating how we can use this nice property in order to design a new kind of high observers for the system considered here and other classes of systems Orlov et al. (2004). We will also investigate a sliding modes observer design for the above class of systems in presence of disturbances Dimassi et al. (2018).

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