Stability properties for two coupled reaction-diffusion equations

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Abstract: This paper studies the stability of the interconnection of two reaction-diffusion equations. We focus on the case where the input and output operators of the interconnection are bounded. Using the spectral decomposition of both equations, we propose a sufficient condition to estimate the exponential stability decay rate of the closed-loop system. This stability test is proposed as constraints of a semidefinite programming. An extension of this condition is also outlined in the form of a Hurwitz criterion. The proposed stability analysis conditions are illustrated with an example of two reaction-diffusion equations with constant couplings terms.

Keywords: Coupled PDEs, Reaction-diffusion equation, Lyapunov analysis, Spectral reduction.

1. INTRODUCTION

Reaction-diffusion phenomena appear in many fields such as pharmaceutics or epidemiology. They belong to the class of parabolic partial differential equations (PDEs). A fundamental problem is to characterize how the trajectories behave, especially to rule on the exponential stability of the steady state.

In the literature, as a particular case of the study of linear operators, the stability analysis of linear PDEs is detailed in Pazy (2012). Modal analysis is the most used approach for studying the stability of a single reaction-diffusion equation. Spectral reduction methods have also led to the synthesis of controllers and observers for single linear parabolic PDE, see for instance Russel (1978); Lasiecka and Triggiani (2000); Morris and Levine (2010); Katz and Fridman (2020). Nevertheless, the case of two interconnected PDEs is more involved: the eigenstructures of the associated linear operator cannot be, in general, computed analytically and do not allow us to assess the stability properties of the plant. Furthermore, by summing the Lyapunov functionals of each PDEs, the application of the Lyapunov theorem leads to conservative stability conditions that are restricted to the case where both PDEs are stable (see Mironchenko and Ito (2015)).

For PDEs interconnections of type hyperbolic-hyperbolic in Di Meglio et al. (2013), hyperbolic-parabolic in Karafyllis and Krstic (2018) or parabolic-parabolic in Grüne and Meurer (2022), the stability analysis methods vary. This diversity comes from the different analysis tools and approximation scheme involved. Global exponential stability (GES) can be assessed via frequency analysis using the small-gain theorem as in Karafyllis and Krstic (2018). It can also rely on the use of Lyapunov functionals which include interconnected terms as in Lhachemi and Prieur (2022). In parallel to this last technique, an approximation is performed to obtain numerical solutions. Finite element methods as in Li et al. (1994); Burman et al. (2022), Legendre polynomials approximation as in Baudouin et al. (2019); Bajodek et al. (2022) or modal decomposition as in Katz and Fridman (2020); Lhachemi and Prieur (2022) can be applied. Then, the application of Lyapunov's theorem leads to linear matrix inequalities (LMIs), which can be solved by linear semidefinite programming (see e.g. Valmorbida et al. (2015); Peet (2021)).

Here, the case of parabolic-parabolic interconnected PDEs is investigated. The proposed methodology consists of

- performing a modal decomposition of the Sturm-Liouville operators;
- projecting the coupling terms on this eigenbasis;
- rewriting the infinite-dimensional interconnected system as a finite-dimensional system coupled with a residual infinite-dimensional dynamics;
- applying Lyapunov's theorem on the finite-dimensional system while evaluating the impact of the residual infinite-dimensional dynamics.

The objective is to derive GES sufficient conditions and provide the exponential decay rate of the solutions.

The paper is structured as follows. Section II introduces the system description and the modal decomposition of the two Sturm-Liouville operators. An equivalent model based on this modal decomposition is proposed. Section III is concerned with the derivation of sufficient stability analysis condition. The first condition, in Theorem 8, is an LMI test and the second one, in Theorem 12, is a Hurwitz test of the approximated state matrix. Lastly, Section IV shows numerically that the coupling of two parabolic PDEs can have a stabilizing effect.

Notation: Throughout this paper, notations $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n_{\succ 0(\prec 0)}$ stand for real numbers, positive numbers, matrices of size $n \times m$ and positive (negative) definite matrices of size n, respectively. Matrices I_n, $M^\top$ and $\mathrm{diag}(u_1, \dots, u_n)$ correspond to the identity matrix of size n, transpose of matrix M in $\mathbb{R}^{n\times n}$ and diagonal matrix whose diagonal coefficients are $(u_1, \dots, u_n)$. We also use $\mathrm{vec}(\cdot)$ as the vectorization operation and $\otimes$ as the

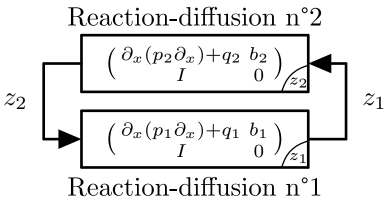

Fig. 1. Block diagram of system (1).

Kronecker product. For any positive matrix M, $\underline{\sigma}(M)$ and $\bar{\sigma}(M)$ stand for its minimal and maximal singular value. Matrix $M \in \mathbb{R}^{n\times n}$ is said to be μ-Hurwitz if $M + \mu I_n$ is Hurwitz and $\mu > 0$. We introduce some functional spaces for real-values functions defined on $[0, 1]$. Denote by C^0 and L^2 the spaces of continuous and square-integrable functions, respectively. Denote also by C^k the set of functions with the k first derivatives in C^0 and by H^k the k-th order Sobolev space. Lastly, the Euclidian norm $|\cdot|$ and the L^2 norm $\|\cdot\|$ are defined by

$$|\cdot| := \begin{cases} \mathbb{R}^n & \mapsto \mathbb{R}_+ \\ u & \to \sqrt{u^\top u} \end{cases}, \quad \|\cdot\| := \begin{cases} L^2 & \mapsto \mathbb{R}_+ \\ z & \to \sqrt{\int_0^1 |z(x)|^2 \mathrm{d}x} \end{cases}.$$

2. REACTION-DIFFUSION SYSTEMS

In this section, we present the system under investigation and propose an equivalent model thanks to a modal based decomposition as in Lhachemi and Prieur (2022).

2.1 Interconnected plant

Consider two reaction-diffusion equations

$$\begin{cases} \partial_t z_i(t,x) = [\partial_x(p_i(x)\partial_x) + q_i(x)] z_i(t,x) + b_i(x) z_{3-i}(t,x), \\ \cos(\theta_{i0}) z_i(t,0) - \sin(\theta_{i0}) \partial_x z_i(t,0) = 0, \\ \cos(\theta_{i1}) z_i(t,1) + \sin(\theta_{i1}) \partial_x z_i(t,1) = 0, \\ z_i(0,x) = z_{i0}(x), \end{cases} \tag{1}$$

for $i \in \{1, 2\}$ and $(t, x) \in \mathbb{R}_+ \times (0, 1)$. The initial conditions z_{i0} are in $L^2((0, 1); \mathbb{R})$, the scalars θ_{ij} are in $[0, \pi/2]$, the diffusion functions p_i are in $C^1((0, 1); \mathbb{R}_+)$ and the both reaction q_i and interconnection b_i functions are in $C^0((0, 1); \mathbb{R})$. The plant (1) is represented in Fig. 1.

Remark 1. The proposed method applies to any couplings of the form $\mathcal{B}_i z_{3-i}$, where $\mathcal{B}_i$ is a linear and bounded operator of $\mathcal{L}(L^2((0, 1); \mathbb{R})$. The method can also be extended to n coupled reaction-diffusion equations.

The operator of system (1) is given by

$$\mathcal{A} = \begin{pmatrix} \partial_x(p_1\partial_x) + q_1 & b_1 \\ b_2 & \partial_x(p_2\partial_x) + q_2 \end{pmatrix}, \tag{2}$$

on the domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_1) \times \mathcal{D}(\mathcal{A}_2)$, where

$$\mathcal{A}_i = \partial_x(p_i\partial_x) + q_i, \tag{3}$$
$$\mathcal{D}(\mathcal{A}_i) = \{f \in H^2((0, 1); \mathbb{R}), \text{ s.t. } {\scriptstyle \cos(\theta_{i0})f(0) - \sin(\theta_{i0})f'(0) = 0 \atop \cos(\theta_{i1})f(1) + \sin(\theta_{i1})f'(1) = 0}\}.$$

Remark 2. In view of (2) and $-\mathcal{A}_i$ being a Sturm-Liouville operator, classical results on the bounded perturbation of generators of C_0-semigroups (see (Pazy, 2012, Chapter 3)) can be readily invoked for assessing and concluding the well-posedness of system (1), yielding solutions in $C^0(\mathbb{R}_+, L^2((0, 1); \mathbb{R}) \times L^2((0, 1); \mathbb{R}))$.

Definition 3. The trivial solution of system (1) is GES with exponential decay rate $\mu > 0$ if there exist $\kappa > 0$ such that the solution satisfies

$$\left\| \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \right\| \leq \kappa e^{-\mu t} \left\| \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix} \right\|, \tag{4}$$

for all t in $\mathbb{R}_+$.

In this paper, the global exponential stability (GES) of the trivial solution $(z_1, z_2) = (0, 0)$ of system (1) is studied.

2.2 Spectral decomposition

Let the two operators $\mathcal{A}_i = \partial_x(p_i\partial_x) + q_i$ be defined on the domains $\mathcal{D}(\mathcal{A}_i)$ for $i \in \{1, 2\}$ as in (3). From the properties of Sturm-Liouville operators, the eigenvalues $\{\lambda_{i,n}\}_{n\geq 1}$ of each operator $\mathcal{A}_i$ are real and ordered such that $\lambda_{i,n+1} < \lambda_{i,n}$ for any integer $n \geq 1$. For each $i \in \{1, 2\}$, $\lambda_{i,n}$ are simple, tend to $-\infty$ as n tends to infinity and satisfy (see, e.g., Orlov (2017)):

$$\lambda_{i,n} \leq -\pi^2(n-1)^2 \underline{p}_i + \bar{q}_i, \tag{5}$$

where $\underline{p}_i = \inf_{x\in[0,1]}(p_i(x))$ and $\bar{q}_i = \sup_{x\in[0,1]}(q_i(x))$.

They are associated to normalized eigenvectors $\{\varphi_{i,n}\}_{n\geq 1}$ that form a Hilbert basis of L^2.

For i in $\{1, 2\}$ and any integer $n_i \geq 1$, consider the sequence $\{\varphi_{i,1}, \ldots, \varphi_{i,n_i}\}$ and define the vectors

$$\Phi_i(x) = [\varphi_{i,1}(x) \ \cdots \ \varphi_{i,n_i}(x)]^\top \in \mathbb{R}^{n_i}. \tag{6}$$

Define also

$$\zeta_i(t) = \int_0^1 \Phi_i(x) z_i(t,x) \mathrm{d}x, \ \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \in \mathbb{R}^{n_1+n_2}, \ \forall t \in \mathbb{R}_+, \tag{7}$$

the vectors of the n_i projections of the state z_i on the eigenfunctions sets $\{\varphi_{i,1}, \ldots, \varphi_{i,n_i}\}$. Define lastly the error state functions

$$\eta_i(t,x) = z_i(t,x) - \Phi_i^\top(x)\zeta_i(t), \ \forall (t,x) \in \mathbb{R}_+ \times (0,1). \tag{8}$$

Remark 4. Note that in the case p_i, q_i are constant functions, the eigenfunctions are known explicitly as trigonometric functions. In the case $p_i(x) = x$ and $q_i(x) = x - \frac{\nu_i}{x}$, we also know that the eigenfunctions are Bessel functions.

Proposition 5. For given integers $n_i \geq 1$, the dynamics of system (1) can be rewritten as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\zeta(t) = A\zeta(t) + \int_0^1 \begin{bmatrix} \Phi_1(x) b_1(x) \eta_2(t,x) \\ \Phi_2(x) b_2(x) \eta_1(t,x) \end{bmatrix} \mathrm{d}x, \\ \partial_t \eta_i(t,x) = [\partial_x(p_i(x)\partial_x) + q_i(x)]\eta_i(t,x) \\ \quad + b_i(x)\left(\Phi_{3-i}^\top(x)\zeta_{3-i}(t) + \eta_{3-i}(t,x)\right) \\ \quad - \Phi_i^\top(x) \int_0^1 \Phi_i(\xi) b_i(\xi) \left(\Phi_{3-i}^\top(\xi)\zeta_{3-i}(t) + \eta_{3-i}(t,\xi)\right) \mathrm{d}\xi, \\ \cos(\theta_{i0})\eta_i(t,0) - \sin(\theta_{i0})\partial_x \eta_i(t,0) = 0, \\ \cos(\theta_{i1})\eta_i(t,1) + \sin(\theta_{i1})\partial_x \eta_i(t,1) = 0, \end{cases} \tag{9}$$

for $i \in \{1, 2\}$ and $(t, x) \in \mathbb{R}_+ \times (0, 1)$, where the matrices are given by

$$A = \begin{bmatrix} \Lambda_1 & B_1 \\ B_2 & \Lambda_2 \end{bmatrix}, \quad \begin{array}{l} \Lambda_1 = \mathrm{diag}(\lambda_{1,1}, \ldots, \lambda_{1,n_1}), \\ \Lambda_2 = \mathrm{diag}(\lambda_{2,1}, \ldots, \lambda_{2,n_2}), \end{array}$$
$$B_1 = \int_0^1 (\Phi_1 b_1 \Phi_2^\top)(x) \mathrm{d}x, \quad B_2 = \int_0^1 (\Phi_2 b_2 \Phi_1^\top)(x) \mathrm{d}x. \tag{10}$$

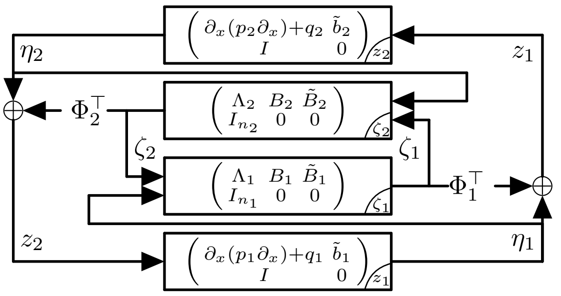

Fig. 2. Block diagram of system (9).

Proof. From one side, the dynamics of the vector ζ are obtained by replacing the state z_i by its decomposition given by (8). From the other side, the dynamics of the error η_i rewrites as follows

$$\begin{aligned}
&\partial_t\eta_i(t,x) = \partial_t z_i(t,x) - \Phi_i^\top(x)\frac{\mathrm{d}}{\mathrm{d}t}\zeta_i(t),\\
&= \partial_t z_i(t,x) - \Phi_i^\top(x)\left(\Lambda_i\zeta_i(t) + \int_0^1 \Phi_i(\xi)b_i(\xi)z_{3-i}(t,\xi)\mathrm{d}\xi\right),\\
&= [\partial_x(p_i(x)\partial_x) + q_i(x)]z_i(t,x) + b_i(x)z_{3-i}(t,x)\\
&\quad -\Phi_i^\top(x)\Lambda_i\zeta_i(t) - \Phi_i^\top(x)\int_0^1 \Phi_i(\xi)b_i(\xi)z_{3-i}(t,\xi)\mathrm{d}\xi,
\end{aligned}$$

which corresponds to the PDE in system (9). Finally, the boundary conditions satisfied by z_i also hold for η_i. ■

The block diagram of this augmented system is represented in Fig 2, where $\tilde{b}_i z_{3-i} := b_i z_{3-i} - \Phi_i^\top \int_0^1 (\Phi_i b_i z_{3-i})(x)\mathrm{d}x$ and $\tilde{B}_i\eta_{3-i} := \int_0^1 (\Phi_i b_i \eta_{3-i})(x)\mathrm{d}x$. It can be seen as an equivalent representation of the original system in Fig.1, where the terms z_i are retrieved using (8).

3. STABILITY TEST

In this section, a GES stability condition for system (1) is derived by the use of the modal augmented model (9). In the following, the time argument will be omitted for simplicity reasons.

3.1 Lyapunov functional

For given integers $n_1, n_2 \geq 1$ and a $P \in \mathbb{S}_{\succ 0}^{n_1+n_2}$, consider the following Lyapunov functional

$$\mathcal{V}(\zeta,\eta_1,\eta_2) = \zeta^\top P\zeta + \|\left[\begin{smallmatrix}\eta_1\\ \eta_2\end{smallmatrix}\right]\|^2, \quad (11)$$

where the augmented state $\zeta = \begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix} \in \mathbb{R}^{n_1+n_2}$ and error state functions $\left[\begin{smallmatrix}\eta_1\\ \eta_2\end{smallmatrix}\right]$ are defined by (7) and (8), respectively. Similarly to Lhachemi and Prieur (2022); Wakaiki and Sano (2020), it is based on the n_i first projections of the state on the orthonormal sequences $\{\varphi_{i,n}\}_{n\geq 1}$ and to the sum of the square L^2 norms of each error functions η_i.

Remark 6. Matrix P can be decomposed as three matrices $P = \begin{bmatrix}P_1 & P_{12}\\ P_{12}^\top & P_2\end{bmatrix}$, where $P_1 \succ 0$, $P_2 \succ 0$ and P_{12} selected to satisfy $P \succ 0$. Note that $\zeta^\top P\zeta$ introduces interconnected terms between the two reaction-diffusion equations thanks to the matrices P_{12}. This is in contrast with Kitsos and Fridman (2022), where $P_{12} = 0$.

Along the trajectories of system (1), using the dynamics expressed by (9) and the orthonormality and completeness of the sequences $\{\varphi_{i,n}\}$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2) &= \zeta^\top PA\zeta + \zeta^\top P\int_0^1 \begin{bmatrix}(\Phi_1 b_1\eta_2)(x)\\ (\Phi_2 b_2\eta_1)(x)\end{bmatrix}\mathrm{d}x\\
&+ \sum_{n=n_1+1}^{\infty}\lambda_{1,n}\left(\int_0^1 \varphi_{1,n}(x)\eta_1(x)\mathrm{d}x\right)^2\\
&+ \int_0^1 \eta_1(x)b_1(x)\left(\Phi_2^\top(x)\zeta_2 + \eta_2(x)\right)\mathrm{d}x \qquad (12)\\
&+ \sum_{n=n_2+1}^{\infty}\lambda_{2,n}\left(\int_0^1 \varphi_{2,n}(x)\eta_2(x)\mathrm{d}x\right)^2\\
&+ \int_0^1 \eta_2(x)b_2(x)\left(\Phi_1^\top(x)\zeta_1 + \eta_1(x)\right)\mathrm{d}x.
\end{aligned}$$

In the following, $\mathcal{V}$ in (11) and its derivatives in (12) will be used to state on the stability of system (1) in the sense of the L^2 norm as in Definition 3.

Remark 7. The stability tests induced by the Lyapunov functional (11) are given in L^2 norm. Note that, considering the following Lyapunov functional

$$\mathcal{V}_{H^1}(\zeta,\eta_1,\eta_2) = \zeta^\top P\zeta + \sum_{i=1}^{2}\sum_{n=n_i+1}^{\infty}\lambda_i\left(\int_0^1 \varphi_{i,n}(x)\eta_i(x)\mathrm{d}x\right)^2, \quad (13)$$

other stability tests could be given in H^1 norm. One can refer to Lhachemi and Prieur (2022) for details.

3.2 Main theorem

The stability of system (1) is studied exploiting (11)-(12) by formulating inequalities that guarantee the decrease of (11) along the trajectories of (1). These inequalities are cast as linear matrix inequalities (LMIs), affine on the parameter P of (11), which can suitably written as constraints of a semidefinite programming.

Theorem 8. For given orders $n_1, n_2 \geq 1$ and scalar $\mu > 0$, if there exist $P \in \mathbb{S}_{\succ 0}^{n_1+n_2}$ and scalars $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$\begin{bmatrix}\Gamma(P) + \begin{bmatrix}\alpha_2\|b_2\Phi_1\|^2 I_{n_1} & 0\\ 0 & \alpha_1\|b_1\Phi_2\|^2 I_{n_2}\end{bmatrix} & P\\ P^\top & \begin{bmatrix}-\beta_1 I_{n_1} & 0\\ 0 & -\beta_2 I_{n_2}\end{bmatrix}\end{bmatrix} \prec 0, \quad (14a)$$

$$\begin{bmatrix}\begin{bmatrix}\gamma_{1,n_1}+\beta_2\|b_2\Phi_2\|^2 & \|b_1\|+\|b_2\|\\ \|b_1\|+\|b_2\| & \gamma_{2,n_2}+\beta_1\|b_1\Phi_1\|^2\end{bmatrix} & I_2\\ I_2 & \begin{bmatrix}-\alpha_1 & 0\\ 0 & -\alpha_2\end{bmatrix}\end{bmatrix} \prec 0, \quad (14b)$$

with $A \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ as in (10) and

$$\Gamma(P) = PA + A^\top P + 2\mu P,$$
$$\gamma_{1,n_1} = 2(-\pi^2 n_1^2 \underline{p}_1 + \bar{q}_1 + \mu),\ \gamma_{2,n_2} = 2(-\pi^2 n_2^2 \underline{p}_2 + \bar{q}_2 + \mu).$$

Then, the trivial solution of system (1) is GES with the exponential decay rate μ.

Proof. Firstly, the Lyapunov functional $\mathcal{V}$ defined by (11) satisfies the following inequalities

$$\begin{aligned}
&\min\{\underline{\sigma}(P),1\}\left(|\zeta|^2 + \|\eta_1\|^2 + \|\eta_2\|^2\right) \leq \mathcal{V}(\zeta,\eta_1,\eta_2),\\
&\mathcal{V}(\zeta,\eta_1,\eta_2) \leq \max\{\bar{\sigma}(P),1\}\left(|\zeta|^2 + \|\eta_1\|^2 + \|\eta_2\|^2\right).
\end{aligned} \quad (15)$$

Then, under the assumptions in the theorem, we show that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2) + 2\mu\mathcal{V}(\zeta,\eta_1,\eta_2) \leq 0$ along the trajectories of system (1). To do so, let us focus on each term of $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2)$ in (12). For the terms in η_i with $i \in \{1,2\}$, the ordering of the eigenvalues and the bound (5) give $\lambda_{i,n} < \lambda_{i,n_i+1} < -\pi^2 n_i^2 \underline{p}_i + \bar{q}_i$ for any $n \geq n_i$ so that

$$\sum_{n=n_i+1}^{\infty} \lambda_{i,n} \left(\int_0^1 \varphi_{i,n}(x)\eta_i(x)\mathrm{d}x \right)^2 \leq (-\pi^2 n_i^2 \underline{p}_i + \bar{q}_i)\|\eta_i\|^2. \tag{16}$$

For the two terms involving η_1 and η_2, applying twice Cauchy-Schwarz's inequality yields

$$\int_0^1 (b_1(x)+b_2(x))\eta_1(x)\eta_2(x)\mathrm{d}x \leq (\|b_1\|+\|b_2\|)\|\eta_1\|\|\eta_2\|, \tag{17}$$

Concerning the crossed term between η_1 and ζ_2, with any positive scalar α_1, Young's inequality leads to

$$\begin{aligned} &\left| 2\int_0^1 (\eta_1 b_1 \Phi_2^\top)(x)\zeta_2 \mathrm{d}x \right| \\ &\quad \leq \frac{1}{\alpha_1}\int_0^1 |\eta_1(x)|^2 \mathrm{d}x + \alpha_1 \int_0^1 |b_1(x)\Phi_2^\top(x)\zeta_2|^2 \mathrm{d}x, \\ &\quad \leq \frac{1}{\alpha_1}\|\eta_1\|^2 + \alpha_1 \|b_1\Phi_2\|^2 |\zeta_2|^2. \end{aligned} \tag{18}$$

Similarly, with any positive scalar α_2, we obtain

$$2\left|\int_0^1 (\eta_2 b_2 \Phi_1^\top)(x)\zeta_1 \mathrm{d}x\right| \leq \frac{1}{\alpha_2}\|\eta_2\|^2 + \alpha_2\|b_2\Phi_1\|^2|\zeta_1|^2. \tag{19}$$

Furthermore, for $i \in \{1,2\}$ with any positive scalars β_i, the application of Cauchy–Schwarz's inequality leads to

$$\beta_i \left| \int_0^1 (\Phi_i b_i \eta_{3-i})(x)\mathrm{d}x \right|^2 \leq \beta_i \|b_i\Phi_i\|^2 \|\eta_{3-i}\|^2. \tag{20}$$

From (12) and (16)-(20), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2) + 2\mu\mathcal{V}(\zeta,\eta_1,\eta_2) \\ &\leq \xi^\top \begin{bmatrix} \Gamma(P) + \begin{bmatrix} \alpha_2\|b_2\Phi_1\|^2 I_{n_1} & 0 \\ 0 & \alpha_1\|b_1\Phi_2\|^2 I_{n_2} \end{bmatrix} & P \\ P^\top & \begin{bmatrix} -\beta_1 I_{n_1} & 0 \\ 0 & -\beta_2 I_{n_2} \end{bmatrix} \end{bmatrix} \xi \\ &\quad + \eta^\top \begin{bmatrix} \gamma_{1,n_1} + \beta_2\|b_2\Phi_2\|^2 + \frac{1}{\alpha_1} & \|b_1\|+\|b_2\| \\ \|b_1\|+\|b_2\| & \gamma_{2,n_2} + \beta_1\|b_1\Phi_1\|^2 + \frac{1}{\alpha_2} \end{bmatrix} \eta, \end{aligned} \tag{21}$$

where $\xi = \begin{bmatrix} \zeta \\ \int_0^1 (\Phi_1 b_1 \eta_2)(x)\mathrm{d}x \\ \int_0^1 (\Phi_2 b_2 \eta_1)(x)\mathrm{d}x \end{bmatrix}$ and $\eta = \begin{bmatrix} \|\eta_1\| \\ \|\eta_2\| \end{bmatrix}$.

If the LMIs (14) hold, then the following inequality holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2) + 2\mu\mathcal{V}(\zeta,\eta_1,\eta_2) \leq 0.$$

Regarding (15) and applying the Lyapunov theorem, it implies that the following inequality holds

$$\left(|\zeta|^2 + \left\|\left[\begin{smallmatrix}\eta_1\\ \eta_2\end{smallmatrix}\right]\right\|^2\right)(t) \leq \frac{\max\{\bar{\sigma}(P),1\}}{\min\{\underline{\sigma}(P),1\}} e^{-2\mu t} \left(|\zeta|^2 + \left\|\left[\begin{smallmatrix}\eta_1\\ \eta_2\end{smallmatrix}\right]\right\|^2\right)(0),$$

Since the augmented state ζ and the errors (η_1,η_2) are related to the state (z_1,z_2) through relations (7) and (8), the orthonormality of the decomposition gives

$$\|z_1(t)\|^2 + \|z_2(t)\|^2 = |\zeta(t)|^2 + \|\eta_1(t)\|^2 + \|\eta_2(t)\|^2. \tag{22}$$

Therefore, the $L^2 \times L^2$ norm of the solution (z_1,z_2) vanishes with an exponential decay rate μ as in Definition 3, which concludes the proof. ∎

Remark 9. In (14) the LMIs have $N_v = \frac{(n_1+n_2)(n_1+n_2-1)}{2} + 4$ variables and $N_c = (n_1+n_2)(n_1+n_2-1)+10$ constraints. Both N_v and N_c depend quadratically on the orders n_1 and n_2. Hence, the computational complexity using the current interior-point solver remains polynomial in n_1 and n_2.

Remark 10. For $(i,j) \in \{1,2\}^2$, the quantities $\|b_i\Phi_j\|^2$ are upper bounded by $\|b_i\|^2\|\Phi_j\|^2 = n_j\|b_i\|^2$, by orthonormality of the sequence Φ_j. At the cost of a loss of conservatism, we can simplify the condition provided in Theorem 8 by replacing the coefficients $\|b_i\Phi_j\|^2$ by $n_j\|b_i\|^2$.

Using the previous remark and imposing $n_1 = n_2 = n$, we simplify the stability condition from Theorem 8 as follows.

Corollary 11. For a given integer $n \geq 1$ and scalar $\mu > 0$, if there exist a $P \in \mathbb{S}^{2n}_{\succ 0}$ and scalars $\alpha, \beta > 0$ such that

$$\begin{bmatrix} \Gamma(P) + \alpha n \bar{b}^2 I_{2n} & P \\ P^\top & -\beta I_{2n} \end{bmatrix} \prec 0, \tag{23a}$$

$$\begin{bmatrix} \gamma_n + 2\bar{b} + \beta n \bar{b}^2 & 1 \\ 1 & -\alpha \end{bmatrix} \prec 0, \tag{23b}$$

with $A \in \mathbb{R}^{2n\times 2n}$ as in (10) and

$$\begin{aligned} \Gamma(P) &= PA + A^\top P + 2\mu P, \\ \gamma_n &= 2\left(-\pi^2 n^2 \min\left\{\underline{p}_1, \underline{p}_2\right\} + \max\{\bar{q}_1,\bar{q}_2\} + \mu\right), \\ \bar{b} &= \max\{\|b_1\|,\|b_2\|\}. \end{aligned}$$

Then, the trivial solution of system (1) is GES with the exponential decay rate μ. Furthermore, if the LMIs (23) hold for order $n+1$, then the conditions hold for order n.

In Theorem 8 and Corollary 11, in order to satisfy the LMIs conditions, the matrix A must be μ-Hurwitz (i.e. the real parts of the eigenvalues of matrix A in (10) are smaller than $-\mu$ with $\mu > 0$).

In the next subsection, this criterion is further explored to derive sufficient conditions of stability in terms of the singular values of the matrix A.

3.3 A simpler criterion for stability

As an extension of Corollary 11, a simple stability test based on the eigenvalues of the approximated matrix A is put forward. It relies on a particular choice of matrix P and scalars α, β with respect to the order n.

Theorem 12. For a given order $n \geq 1$, assume that A in (10) with $n_1 = n_2 = n$ is μ-Hurwitz. If $\bar{\mu} > 0$ holds where

$$\bar{\mu} = \mu - \frac{n^{\frac{3}{2}}\bar{b}}{\Gamma_n + \sqrt{\Gamma_n^2 + \frac{n^2}{2}}}, \tag{24}$$

$$\Gamma_n = \left(\pi^2 n^{\frac{5}{2}} \frac{\min\{\underline{p}_1,\underline{p}_2\}}{4\bar{b}} - n^{\frac{1}{2}} \frac{\max\{\bar{q}_1,\bar{q}_2\}+\mu+\bar{b}}{4\bar{b}}\right),$$

then, the trivial solution of system (1) is GES for any exponential decay rate $\mu' < \bar{\mu}$.

Proof. For any $W \succ 0$, consider the symmetric solution P_0 of the Lyapunov equation

$$P_0 A + A^\top P_0 + 2\mu P_0 = -W,$$

which is a positive definite matrix. Assuming that A is μ-Hurwitz, the Lyapunov matrix P_0 is given by

$$P_0 = -\mathrm{vec}^{-1}\left((A^\top \otimes I_{2n} + I_{2n} \otimes A^\top + 2\mu)^{-1}\mathrm{vec}(W)\right). \tag{25}$$

For any $\rho > 0$, taking the variables

$$P = \frac{1}{\bar{\sigma}(P_0)} P_0, \quad \alpha = \frac{\sqrt{\rho}}{\bar{b}} n^{-\frac{3}{2}}, \quad \beta = \frac{1}{\sqrt{\rho}\bar{b}} n^{\frac{1}{2}}, \tag{26}$$

we note that

$$\mu' - \mu + 2\sqrt{\rho} n^{-\frac{1}{2}} \bar{b} < 0, \tag{27a}$$

$$-\pi^2 n^2 \min\{\underline{p}_1, \underline{p}_2\} + \max\{\bar{q}_1, \bar{q}_2\} + \mu' + \bar{b} + \frac{n^{\frac{3}{2}}\bar{b}}{\sqrt{\rho}} < 0, \tag{27b}$$

imply by Schur complement that the LMIs (23a)-(23b) hold. In order to make identical conditions (27a)-(27b), we select a suitable scalar $\rho > 0$ such that

$$\rho + \left(\pi^2 n^{\frac{5}{2}} \frac{\min\{\underline{p}_1, \underline{p}_2\}}{2\bar{b}} - n^{\frac{1}{2}} \frac{\max\{\bar{q}_1, \bar{q}_2\} + \mu + \bar{b}}{2\bar{b}}\right)\sqrt{\rho} - \frac{n^2}{2} = 0,$$

or equivalently

$$\sqrt{\rho} = -\Gamma_n + \sqrt{\Gamma_n^2 + \frac{n^2}{2}} = \frac{\frac{n^2}{2}}{\Gamma_n + \sqrt{\Gamma_n^2 + \frac{n^2}{2}}}. \tag{28}$$

Then, we obtain the two conditions (27a)-(27b) if

$$\mu' < \mu - 2\sqrt{\rho} n^{-\frac{1}{2}} \bar{b} = \mu - \frac{n^{\frac{3}{2}}\bar{b}}{\Gamma_n + \sqrt{\Gamma_n^2 + \frac{n^2}{2}}} = \bar{\mu},$$

with the particular choice of P, α, β in (26) and ρ in (28). Consequently, the LMIs (23a)-(23b) are satisfied for any $\mu' < \bar{\mu}$. Assuming now that $\bar{\mu} > 0$, Corollary 11 guarantees GES and decay rates $\mu' \in (0, \bar{\mu})$ for system (1). ∎

Theorem 12 provides a sufficient stability condition and guarantees exponential decay rates for system (1) in terms of an eigenvalue test on matrix A in (10). Note that such guaranteed decay rates are lower bounds of the exact decay rate of the system. Indeed, for a given order n, Theorem 12 is more conservative than Theorem 8 and Corollary 11. Nevertheless, the eigenvalue test may be faster and more appropriate than the LMI test.

As the order n goes to infinity, this theorem can be then rewritten as the asymptotic condition stated below. Similar asymptotic results have been obtained in Katz and Fridman (2020) regarding the synthesis of controllers and observers for a single reaction-diffusion equation.

Corollary 13. If the matrix A in (10) with $n_1 = n_2 = n$ is μ-Hurwitz when n tends to infinity, then the trivial solution of system (1) is GES for any exponential decay rate $\mu' < \mu$.

Proof. The proof is directly linked to the passage to the limit in Theorem 12. Indeed, (24) becomes

$$\bar{\mu} = \mu - \frac{n^{\frac{3}{2}}\bar{b}}{\Gamma_n + \sqrt{\Gamma_n^2 + \frac{n^2}{2}}} \underset{n\to\infty}{\longrightarrow} \mu,$$

because $\Gamma_n \underset{n\to\infty}{\sim} n^{\frac{5}{2}}$ Thus, the scalar $\bar{\mu}$ can be replaced asymptotically by μ. ∎

Corollary 13 ensures a bound for the exponential decay rate of system (1), although it may be pessimistic.

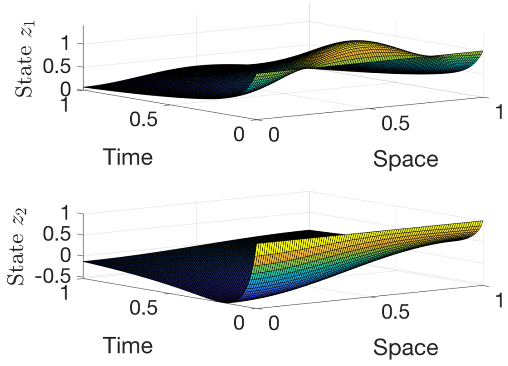

Fig. 3. Solution $(z_1, z_2)(t)$ for $(z_1, z_2)(0) = 1$.

4. EXAMPLE

In this section, the proposed sufficient conditions of stability are illustrated. We show that two stable reaction-diffusion systems can result in an unstable interconnection under the effect of the input and output operators.

Consider the following example

$$\begin{cases} \partial_t \begin{bmatrix} z_1(t,x) \\ z_2(t,x) \end{bmatrix} = \begin{bmatrix} p_1\partial_{xx} + q_1 & b_1\cos(\pi x) \\ b_2\cos(\pi x) & p_2\partial_{xx} + q_2 \end{bmatrix} \begin{bmatrix} z_1(t,x) \\ z_2(t,x) \end{bmatrix}, \\ \partial_x z_1(t,0) = 0, \quad \partial_x z_1(t,1) = 0, \\ \partial_x z_2(t,0) = 0, \quad \partial_x z_2(t,1) = 0, \end{cases} \tag{29}$$

where p_1, p_2, q_1, q_2 are constant functions and given by

$$q_1 = q_2 = 1, \quad p_1 = p_2 = 2, \quad b_1 = -b_2 = 10. \tag{30}$$

In that case, we know that the Sturm-Liouville diagonal operators $-(p_i\partial_{xx} + q_i)$ are associated to the eigenvectors $\{\varphi_{i,n}\}_{n\geq 1}$ and eigenvalues $\{\lambda_{i,n}\}_{n\geq 1}$ with

$$\begin{aligned} \varphi_{i,n}(x) &= \begin{cases} 1, \text{ if } n = 1, \\ \sqrt{2}\cos((n-1)\pi x), \text{ otherwise}, \end{cases} \\ \lambda_{i,n} &= -p_i((n-1)\pi)^2 + q_i. \end{aligned} \tag{31}$$

We also know that $\bar{b} = \frac{1}{\sqrt{2}}\max\{b_1, b_2\}$. Furthermore, for any integers $n_1 = n_2 = n \geq 1$, the matrix A in (10) is given by

$$\begin{aligned} A &= \begin{bmatrix} \Lambda_1 & b_1 B \\ b_2 B^\top & \Lambda_2 \end{bmatrix}, \\ \Lambda_1 &= -p_1 \mathrm{diag}(1, \dots, n^2)\pi^2 + q_1, \\ \Lambda_2 &= -p_2 \mathrm{diag}(1, \dots, n^2)\pi^2 + q_2, \\ B &= \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Both equation taken separately are unstable but make a stable system once interconnected. For instance, a stable solution is simulated in Fig. 3 from the initial condition $z_1(0,x) = z_2(0,x) = 1$ for all $x \in (0,1)$.

Indeed, applying Theorem 8, the LMIs conditions (14) with $\mu = 10^{-5}$ are satisfied for orders (n_1, n_2) such that $n_1 \geq 2$ and $n_2 \geq 2$. Applying Corollary 11 with $\mu = 10^{-5}$, the LMIs conditions (14) are also satisfied for order $n \geq 2$. The second version provides a gain in

Order n	1	2	5	10	15	20	30
Decay μ	–	0.001	1.19	1.55	1.67	1.73	1.78
CPU time	0.1s	0.1s	0.2s	0.3s	0.6s	2s	6s

Table 1. Exponential decay rate μ guaranteed by Corollary 11 for a given order n.

Order n	≤ 5	6	10	20	50	100	500
Decay $\bar{\mu}$	–	0.19	0.89	1.40	1.71	1.81	1.89
CPU time	0.01s	0.01s	0.01s	0.02s	0.05s	0.1s	0.2s

Table 2. Exponential decay rate $\bar{\mu}$ guaranteed by Theorem 12 for a given order n.

terms of computational complexity since it avoids testing pairs (n_1, n_2) by imposing $n_1 = n_2 = n$. With a bisection algorithm, it is even possible to find the maximal decay rate μ which can be guaranteed by Corollary 11. The results as well as the computational times are summarized in Table 1.

Moreover, applying Theorem 12, the eigenvalue test on the matrix A is performed for several orders n. Starting from the scalar μ such that A is μ-Hurwitz, we remove the error gap $\frac{n^{\frac{3}{2}}\bar{b}}{\Gamma_n+\sqrt{\Gamma_n^2+\frac{n^2}{2}}}$ to obtain the certified decay rate $\bar{\mu}$. For $n = 1$, the matrix A is not Hurwitz and we cannot conclude on stability properties. For $n \in [2, 5]$, even if A is Hurwitz by construction, we end up with $\bar{\mu} < 0$ and do not assess stability. For $n \geq 6$, the scalar $\bar{\mu}$ is positive allowing us to state that the system is GES, with at least a decay rate given by $\bar{\mu}$. The results are shown in Table 2. Interestingly, the exponential decay rate $\bar{\mu}$ guaranteed by Theorem 12 increases with the order n. We finally note that A is 1.91-Hurwitz when $n \to \infty$. According to Corollary 13, this system is GES and has at least an exponential decay rate of 1.91.

Lastly, comparing Table 1 and Table 2 for a fixed certified decay rate, we note that using an eigenvalue test instead of a LMI test reduces the calculation time drastically.

5. CONCLUSIONS

This paper has discussed the stability analysis of coupled reaction-diffusion systems. With the help of the modal decomposition of each subsystem, a sufficient condition of stability has been proposed. Then, a simple test on the eigenvalues of the approximated state matrix has also been highlighted. In the numerical example section, we were able to conclude that stability properties of reaction-diffusion systems can be gained through interconnection.

Future work will investigate interconnections with unbounded input and output operators. It would also be interesting to focus on nonlinear reaction-diffusion equations and determine the basin of attraction of a stable equilibrium, expressed with a finite number of state variables.

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