A Comprehensive Framework to Determine Lyapunov Functions for a Set of Continuous Time Stability Problems

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***Abstract*—Analyzing the stability of nonlinear systems, with or without external inputs, is still a very challenging task. An efficient way consists in using the Lyapunov theory which can be deployed by searching for a Lyapunov function satisfying constraints depending on the considered stability problem. By showing that a majority of stability problems can be deduced from the Input-to-State Stability (ISS) one, a generic framework is proposed for tackling a catalog of stability analysis problems for continuous time systems. More precisely, it allows the user to determine Lyapunov functions, whose structure can be arbitrarily chosen, and that maximizes the size of the corresponding estimation of the Domain Of Attraction (DOA). This generic work is illustrated by ISS example showing the efficiency of the approach.**

***Index Terms*—Lyapunov theory, Input-to-State stability, optimization, nonlinear systems, genetic algorithm.**

I. INTRODUCTION

Lyapunov theory, introduced in the late nineteenth century [1], is a practical way to investigate for the stability of an equilibrium point for a dynamical system. The method relies on searching for a function (called Lyapunov function) that exhibits four important properties in the case of continuous time systems that are sufficient for establishing stability on a Domain Of Attraction (DOA) of a stable equilibrium point : (1) it must be a local positive definite function; (2) it must have continuous partial derivatives; (3) its time derivative along any state trajectory starting from the DOA must be negative semi-definite; and (4) the function tends to infinity whatever the direction in the state space.

The general problem of constructing a Lyapunov function in order to maximize the size of a guaranteed stability region included in the actual unknown DOA is difficult. There have been numerous attempts and methods in the literature on how to compute Lyapunov functions for various kinds of systems in order to achieve a stability result. Some of them use physical insights into the system to have a good intuition about a candidate for a Lyapunov function, for instance using some energy concepts leading to quadratic forms. Although more conservative in this form, those Lyapunov functions have a physical meaning and their computation are simpler. Other approaches allow more generic Lyapunov functions to be obtained, using modern optimization tools or machine learning techniques. In spite of a reduced physical meaning, those techniques are less conservative but are ensuring, which is the most important in our opinion, a larger estimation of the DOA.

Attempts have been made to compute quadratic Lyapunov function maximizing the estimate of the DOA. One of the first such publication is [2] where authors minimize the product of the eigenvalues of $P > 0$ in $V(x) = x^T Px$. In [3], the authors are looking for the best quadratic Lyapunov function using adaptive tabu-search. The best quadratic function is the one which ensures the largest DOA for this form of Lyapunov function. Besides, [4] proposed to compute a quadratic Lyapunov function which maximizes the volume of the DOA using Linear Matrix Inequalities for polynomial nonlinear systems. The reader can have a wide overview of existing methods for computing such Lyapunov functions in [5]. These methods have good assets : first, there is a physical insight behind this and a link between the quadratic Lyapunov function and the energy laws of a system can be done. Moreover, the calculation time remains quite reasonable. However, from our point of view, these methods may become too conservative to be usable in case of industrial complex systems to estimate a sufficiently large DOA. Thus, it could be interesting to have easy-to-use tools for computing Lyapunov functions less depending on their form and with less assumptions about the form of the system.

In the paper [6], the authors propose an interesting and promising approach for the construction of a Lyapunov function which is modeled by a neural network using the well-know universal approximation capability of neural networks. Although effective, the main difficulty of this approach is related to the absence of the maximisation of the DOA and the use of barrier functions, which do not ensure that the final result is a Lyapunov function : a post verification is required. Besides, [7] suggests a symbolic regression strategy for computing Lyapunov functions for nonlinear systems. However, in this paper, the conservatism of the Lyapunov function depends on the function library that the user has to chose. In addition, more complex stability problems are not tackled, such as exponential stability or ISS.

In [8], we propose to use a new constrained optimization scheme to determine a Lyapunov function modeled by a neural network while maximizing the domain of attraction. Indeed, the weights of this neural network are calculated in a way that is mathematically proven to result in a Lyapunov function. The authors made an extension for discrete time cases in [9].
However, the approach appears to be quite generic and in fact not only dedicated to neural networks. In fact, it could be a good politics to allow user to use any form for Lyapunov functions provided that they are differentiable. Indeed, if the system is not too complex, determining a quadratic Lyapunov function can be a good choice to gain some computation time. On the contrary, complex industrial systems need more complex Lyapunov functions as the ones modeled by a neural network.

Thus, in this paper, we propose to generalize the proposed method of [9] by a generic framework to determine a Lyapunov function : the Lyapunov function can be modeled by any differentiable function and a variety of stability problems are tackled : Lyapunov stability, asymptotic stability, exponential stability and Input-to-State stability. Indeed, by addressing the ISS problem, we show, thanks to the form of the proposed constrained optimization scheme, that the other types of stability problems can be easily derived from the ISS case.

In this work, we propose a unified framework to determine Lyapunov functions. Our main contributions rely on the following aspects :

- We propose an efficient and generic constrained optimization scheme to generate a Lyapunov function and prove the positiveness and definitiveness properties.
- We show that treating different stabilities (from ISS to basic stability) can be handled by adding constraints to the algorithm, and requires no additional tuned-by-hand parameters compared with the initial method.
- We provide a comprehensive method, allowing a user-friendly approach, to generate a Lyapunov function for several classes of systems.

Note that the goal of this study is not to stabilize a given plant, but to analyze its stability, by computing automatically an estimation as large as possible of the DOA. This paper does not deal of the stabilization with the system itself.

The paper is composed of 3 sections : the first one reminds some key features related to the Lyapunov theory. The second one proposes a generic approach to compute a Lyapunov function maximizing the DOA. Starting from the ISS problem, it is shown how to extend it easily to other simpler stability problems. Finally, we provide one example which deals with Lyapunov function modeled by a neural network and Input-to-State stability.

II. KEY FEATURES ABOUT THE LYAPUNOV THEORY

A. Lyapunov Theory for Continuous Time Systems

In this section, we introduce notations and definitions, and present some key results required for the development of the main contribution of this paper. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\|.\|$ denotes a norm on $\mathbb{R}^n$, and $\mathbb{X} \subset \mathbb{R}^n$, a set containing $X = 0$.

Consider the autonomous system given by (1).

$$\dot{X} = f(X) \tag{1}$$

where $f : \mathbb{X} \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $\mathbb{X} \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and there is at least one equilibrium point X_e, that is :

$$f(X_e) = 0. \tag{2}$$

Theorem 1 [10]. Let $X_e = 0$ be an equilibrium point for (1) and a set D containing 0 in its interior is the domain of attraction. If there exist a continuously differentiable function V such that :

$$V(0) = 0 \text{ and } V(X) > 0 \text{ in } D - \{0\} \tag{3}$$

$$\dot{V}(X) \leq 0 \text{ in } D \tag{4}$$

then, $X_e = 0$ is Lyapunov stable, where $D \subset \mathbb{X} \subset \mathbb{R}^n$ is called the Domain Of Attraction (DOA). Qualitatively, the larger the DOA is, the more disturbances can be accommodated that will "move" the system from its equilibrium point.

For readability, we note $G(X) = \dot{V}(X)$.

Note that if 0 is an equilibrium point, then $G(0)$ has to be equal to 0.

Finally if,

$$G(X) < 0 \text{ in } D - \{0\} \tag{5}$$

then, $X_e = 0$ is asymptotically stable.

This means that the system will converge to 0 from every initial point X_0 belonging to D.

Outside D, one can not conclude anything about the stability. In this paper, the continuous case will be presented, but the discrete time side can be easily obtained from the work in [9].

B. A Catalog of Stability Problems

A list of stability properties achieved in this paper is showed here.

Exponential Stability

Theorem 2 [10]. Let $X_e = 0$ be an equilibrium point for (1). Let $V : D \rightarrow \mathbb{R}$ be a continuous function such that [10] :

$$0 < k_1||X||^p \leq V(X) \leq k_2||X||^p \text{ in } D - \{0\} \tag{6}$$

$$G(X) \leq -k_3||X||^p < 0 \text{ in } D - \{0\} \tag{7}$$

where k_1, k_2, k_3 and p are positive constants, then $X_e = 0$ is exponentially stable.

Note : $\lambda < \frac{k_2}{k_3}$ is an upper-bound of the exponential convergence rate of $||X(t)||$ [11].

Input-to-State stability

Consider the system given by (8).

$$\dot{X} = f(X, \Gamma) \text{ with } \Gamma \text{ an exogenous signal} \tag{8}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a locally Lipschitz map from a domain $X \in \mathbb{X} \subset \mathbb{R}^n$, $\Gamma \in \mathbb{L} \subset \mathbb{R}^m$ into $\mathbb{R}^n$ and there is at least one equilibrium point $X_e = 0$, for the autonomous system, i. e. $\Gamma = 0$.

Theorem 3 [10]. Let $V : D \rightarrow \mathbb{R}$ be a continuous function such that :

$$\alpha_1(||X||) \leq V(X) \leq \alpha_2(||X||) \tag{9}$$

$$G(X) \leq -\alpha_3(||X||) + \alpha_4(||\Gamma||) \tag{10}$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are $\mathcal{K}_\infty$, then $X_e = 0$ is Input-to-State stable. Remind that a function $h(||X||)$ is $\mathcal{K}_\infty$ if h is a scalar increasing function, $h(0) = 0$ and $h_\infty \rightarrow +\infty$. Thus, $\alpha_i(||X||) = c_i||X||^{b_i}$ are suitable function ($c_i > 0$, $b_i > 0$) according to [10]. In the case where $\Gamma \in \mathcal{L}_2$, i. e. bounded energy, one may be interested to the Integral ISS (iISS) which can be analyzed by enforcing $\alpha_3 : [0, \infty) \rightarrow [0, \infty)$. In the ISS framework, $G(X)$ is written :

$$G(X) = \frac{\partial V}{\partial x} f(x, u) \tag{11}$$

Note : According to [10], ISS stability requires Lyapunov stability when $\Gamma = 0$.

III. PROPOSED GENERIC FRAMEWORK

A. Introduction

In the following, we assume that $X_e = 0$ is an equilibrium point. According to section II, every stability problems (from simple stability case to the ISS case) implies $V(X) > 0$ and $G(X) < 0$. Stability problems are just different regarding constraints on bounds for $V(X)$ and $G(X)$. Thus, to tackle any stability problem, a necessary condition for the searched Lyapunov function $V(X)$ is that :

- In the DOA, $V(X)$ has a local minimum in 0 and $V(0) = 0$.
- In the DOA, $G(X)$ has a local maximum in 0 and $G(0) = 0$.

Sufficient conditions for $V(X)$ to have a local minimum at 0 are :

(v1) $V(0) = 0$.

(v2) $\left.\frac{\partial V}{\partial x_j}\right|_{X=0} = 0$ for all j=1,2,...,n.

(v3) H^V (the matrix of 2nd derivatives of V at $X = 0$) is positive definite.

In the same way, sufficient conditions for $G(X)$ to have a local maximum at 0 are :

(d1) $G(0) = 0$.

(d2) $\left.\frac{\partial G}{\partial x_j}\right|_{X=0} = 0$ for all j=1,2,...,n.

(d3) H^G (the matrix of 2nd derivatives of G at $X = 0$) is negative definite.

In the next section, a generic formula for H^{V} and H^{G} is computed.

In the following, we assume that the user has chosen a structure for $V(X)$ which is twice differentiable : polynomial form, neural network form, quadratic form, network of radial basis form etc. Even if the approach is well-suited to differentiable systems, one can extend it to several other classes of systems (potentially non differentiable) if they are previously identified as such. For instance, saturations, absolute values, sign functions, etc. can be approximated by efficient regression technics.
We note $X = [x_1,...,x_n]^T$ and the Jacobian matrix of f is denoted :

$$F^J := \{J_{qr},\ q = 1, ..., n \text{ and } r = 1, ..., n.\} \tag{12}$$

where :

$$J_{qr} = \left.\frac{\partial f_q}{\partial x_r}\right|_{X=0} \tag{13}$$

B. Expression of H^V

The second derivative of V can be straightforwardly obtained :

$$H^V := \{V_{qr},\ q = 1, ..., n \text{ and } r = 1, ..., n.\} \quad (14)$$

where :

$$V_{qr} = \left.\frac{\partial^2 V}{\partial x_q \partial x_r}\right|_{X=0} \quad (15)$$

C. Expression of H^G

By definition,

$$G(X) = \frac{dV}{dt} = \sum_{h=1}^{n} \frac{\partial V}{\partial x_h} f_h(X) \quad (16)$$

And then, with $l = 1, ..., n$:

$$\frac{\partial G}{\partial x_l} = \sum_{h=1}^{n} \frac{\partial^2 V}{\partial x_h \partial x_l} f_h(X) + \sum_{h=1}^{n} \frac{\partial V}{\partial x_h} \frac{\partial f_h}{\partial x_l} \quad (17)$$

In the case where 0 is an equilibrium point, the following property holds :

$$\left.\frac{\partial G}{\partial x_l}\right|_{X=0} = 0 \quad (18)$$

The second derivative of G is as follows :

$$\begin{aligned} \frac{\partial^2 G}{\partial x_l \partial x_p} &= \sum_{h=1}^{n} \frac{\partial^3 V}{\partial x_h \partial x_l \partial x_p} f_h(X) \\ &+ \sum_{h=1}^{n} \frac{\partial^2 V}{\partial x_h \partial x_l} \frac{\partial f_h}{\partial x_p} + \sum_{h=1}^{n} \frac{\partial^2 V}{\partial x_h \partial x_p} \frac{\partial f_h}{\partial x_l} \\ &+ \frac{\partial V}{\partial x_h} \frac{\partial^2 f_h}{\partial x_l \partial x_p} \end{aligned} \quad (19)$$

with $l = 1, ..., n$ and $p = 1, ..., n$.

We denote by :

$$H^G := \{G_{lp},\ l = 1, ..., n \text{ and } p = 1, ..., n.\} \quad (20)$$

where :

$$G_{lp} = \left.\frac{\partial^2 G}{\partial x_l \partial x_p}\right|_{X=0} \quad (21)$$

Taking into account (2) and (v2), (19) results in :

$$G_{lp} = \sum_{h=1}^{n} V_{hl}\ J_{hp} + \sum_{h=1}^{n} V_{hp}\ J_{hl} \quad (22)$$

Note that the discrete time case can be found in [9].

D. Domain Of Attraction

In this section, the Input-to-State stability DOA maximization problem is addressed. It is reminded (see section III.A) that other stability problems can then be derived from that case.

DOA Maximization Problem

In the following, we keep the notations established in section II.A. In order to tackle the ISS problem (see section II.B), one assumes that the autonomous system is stable, according to [10]. Thus, the searched Lyapunov function has to satisfy (v1)-(v3) and (d1)-(d3) (local minimum for V and local maximum for G at $X = 0$).

Consider $\mathcal{Z}$ a set of points obtained from a hypercube whose faces are gridded in order to cover a sufficiently large enough included in $\subset\ \mathbb{X}\times\mathbb{L}$.

We denote P = max ($ratio_v$, $ratio_{dv}$) where $ratio_v$ are the number of points $(X,\ \Gamma) \subset \mathcal{Z}$ where the condition $\alpha_2(||X||) < V(X) < \alpha_1(||X||)$ is not satisfied and $ratio_{dv}$ are the number of points $(X,\ \Gamma) \subset \mathcal{Z}$ where the condition $G(X) > -\alpha_3(||X||) + \alpha_4(||\Gamma||)$ is not satisfied. In order to maximize the estimated DOA, we have to minimize P.

Optimization Scheme

We assume that a suitable structure for $V_\theta(X)$ has been chosen where all the parameters that aim to characterize the Lyapunov function are collected into θ.

According to section II.B, one searches for functions $\alpha_1, \alpha_2, \alpha_3$ and α_4 belonging to $\mathcal{K}_\infty$. Then, the function $\alpha_i(x)$ can have the following form :

$$\alpha_i(x) = c_i||x||^{b_i},\ c_i > 0 \text{ and } b_i > 0. \quad (23)$$

Then, all searched parameters are collected into Θ where :

$$\Theta = [\theta,\ c_i,\ b_i,\ i = 1, ..., 4]. \quad (24)$$

For an appropriate Lyapunov function to be determined, conditions (v1)-(v3) and (d1)-(d3) need to be satisfied. To this end, a fitness function, $\mathcal{Q}$, should be selected so that positiveness and negativeness respectively of H^V and H^G are constrained. The symmetric matrix H^G is negative definite if all its eigenvalues are negative.

Constrained Implementation

Denote λ_i^v, i=1,...,n the set of the n eigenvalues of H^V and λ_i^g, i=1,...,n the set of the n the eigenvalues of H^G.

The problem can be expressed in the general form of an optimization problem in which the cost function $\mathcal{Q}$ needs to be minimized.

To this purpose, we define :

$H^{V\prime} = H^V \times -1$

Denote $\lambda^{v\prime}$ the eigenvalues of $H^{V\prime}$,

$\bar{\lambda}^{v\prime}$ = max(real($\lambda^{v\prime}$)),

$\bar{\lambda}^g$ = max(real(λ^g)) and

$\bar{\lambda}$ = max ($\bar{\lambda}^{v\prime}$, $\bar{\lambda}^g$)

Then, the fitness function to be minimized has the following form :

$$\min_{\Theta}\ \mathcal{Q}$$

If $\bar{\lambda} \geq 0$, $\mathcal{Q} = \bar{\lambda}$

Else, $\mathcal{Q}$ = - $\dfrac{1}{P+1}$

Note that the definition of $\mathcal{Q}$ can not be singular, even if $P = 0$ or $\alpha_4 = 0$.

Adaptation to the other stability problems

For an exponential stability problem :

- $\mathcal{Z}$ is reduced to $\mathbb{X}$.
- Θ is reduced to $[\theta, c_i, b_i,\ i = 1, ..., 3]$.
- An additional constraint related to the rate of convergence of $||X||$ can be enforced according to section II.B.

For an asymptotic stability problem :

- $\mathcal{Z}$ is reduced to $\mathbb{X}$.
- Θ is reduced to θ.

For Lyapunov stability problem :

- $\mathcal{Z}$ is reduced to $\mathbb{X}$.
- Θ is reduced to θ.
- *ratio*$_{dv}$ are the number of points $X \subset \mathcal{Z}$ where $G(X) > 0$ is evaluated.

IV. Example

In this section, we apply our approach to a nonlinear continuous system to prove ISS with a Lyapunov function modeled by a neural network.
The entire test was performed on a machine equipped with an Intel Core i5 - 8400H (2.5 GHz) processor and 16 GB RAM.
In order to minimize $\mathcal{Q}$, the Genetic Algorithm (GA) from the Global Optimization Toolbox in Matlab is used. We keep all the defaults settings of the GA.

Let us consider the following system :

$$\begin{cases} \dot{x}_1 = -tan(x_1) + x_2^2 \\ \dot{x}_2 = -x_2 + x_1 u^2 \end{cases}$$

The ranges for x_1, x_2 and u are $x_1 \in [-1.5, 1.5]$, $x_2 \in [-1.5, 1.5]$ and $u \in [0, 4]$.

Parameters settings

The parameters settings used in this example are as follows :

- We consider 1 hidden layer and the number of neurons of this hidden layer is arbitrarily set to *K* = 12.
- We set $\mathbb{X}$ as a rectangle of 21 × 21 × 21 points centered at 0. Therefore, the conditions $\alpha_2(||X||) < V(X) < \alpha_1(||X||)$ and $G(X) > -\alpha_3(||X||) + \alpha_4(||\Gamma||)$ are evaluated in 9 261 points in the range of the system.
- The number of searched variables Θ (38) is : K × (2n) + 1 + 4×2 = 57. The searched parameters relative to V ∈ [-4,4] and the parameters relative to α_i functions ∈ [0,1000].

The Lyapunov Function

In this example, we assume that the Lyapunov function $V(X)$ is represented by a neural network where the x_i are the inputs, w_{ji} are the weights of the hidden layer, a_i the weights of the output layer, h_i are the biases of the hidden layer, β is the bias of the output layer; *i*=1,...,*n* and *j*=1,...,*K* where *K* is the number of neurons of the hidden layer and σ is the activation function of the neural network. Here, $\sigma(\nu)$ = tanh(ν) is chosen.

Therefore, $V(X)$ can be expressed as :

$$V(X) = \sum_{i=1}^{K} a_i \sigma(\nu_i) + \beta \quad (25)$$

$$\nu_i = \sum_{j=1}^{n} w_{ji} x_j + h_i \quad (26)$$

The second derivative of *V*(*X*) and *G*(*X*) are computed as functions of the neural network in order to express H^V and H^G :

$$\begin{aligned} V_{qr} &= \left.\frac{\partial^2 V}{\partial x_q \partial x_r}\right|_{X=0} \\ &= \sum_{i=1}^{K} a_i \left.\frac{d^2\sigma(\nu_i)}{d\nu_i^2}\right|_{X=0} \left.\frac{\partial v_i}{\partial x_r}\right|_{X=0} w_{qi} \\ &= \sum_{i=1}^{K} a_i \left.\frac{d^2\sigma(\nu_i)}{d\nu_i^2}\right|_{X=0} w_{ri} w_{qi} \end{aligned} \quad (27)$$

$$\begin{aligned} G_{lp} &= \left.\frac{\partial^2 G}{\partial x_l \partial x_p}\right|_{X=0} \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{K} a_i \left.\frac{d^2\sigma(\nu_i)}{d\nu_i^2}\right|_{X=0} w_{ji} w_{li} \right) J_{jp} + \\ &+ \sum_{j=1}^{n} \left(\sum_{i=1}^{K} a_i \left.\frac{d^2\sigma(\nu_i)}{d\nu_i^2}\right|_{X=0} w_{ji} w_{pi} \right) J_{jl} \end{aligned} \quad (28)$$

where *q*=1,...,*n*; *r*=1,...,*n*; *l*=1,...,*n*; *p*=1,...,*n*.

Assumptions (v2) and (d2) imply the following 2 constraints directly satisfied :

$$\sum_{i=1}^{K} a_i \sigma(h_i) + \beta \ = 0 \quad (29)$$

$$\sum_{i=1}^{K} a_i (1 - tanh^2(h_i)) w_{qi} = 0 \quad \text{for} \quad q=1,...,n. \quad (30)$$

Elements of demonstration can be found in [8].

Results

First, the asymptotic stability of the origin with $u = 0$ is proven, which is a required condition according to [10].
Therefore, we can check the Input-to-State stability. The ISS of the origin is considered.

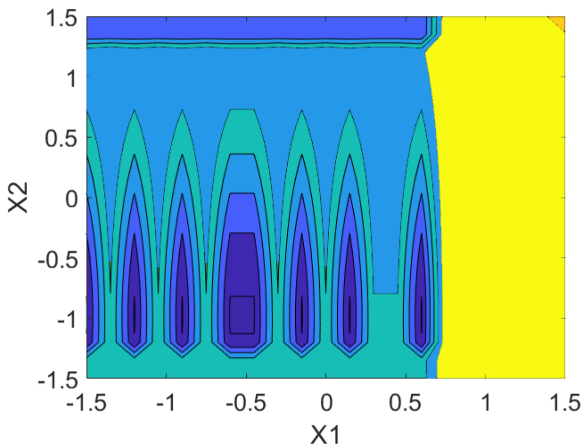

Fig. 1: The estimated DOA for this system.

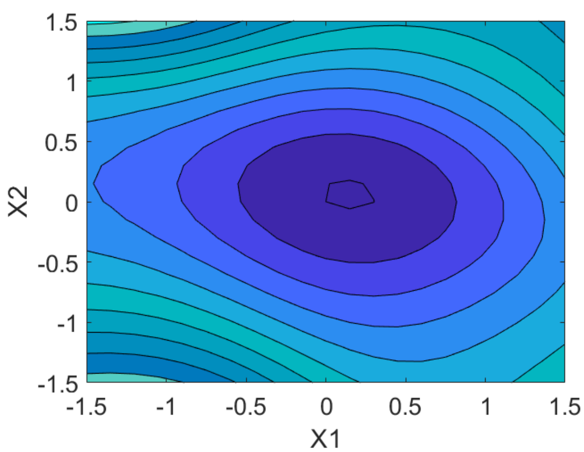

Fig. 2: The estimated DOA for this system.

In the Fig. 1, we can consider the estimated DOA when $u = 4$. The blue/green area represent the region where the origin is ISS, and the area yellow/orange where the origin is not stable. In the Fig. 2, we can consider the estimated DOA when $u \in]0; 4[$. The blue/green area represent the region where the origin is ISS.
In order to show the validity of the results, one uses Simulink with different starting points belonging or not to the DOA in Fig. 3.

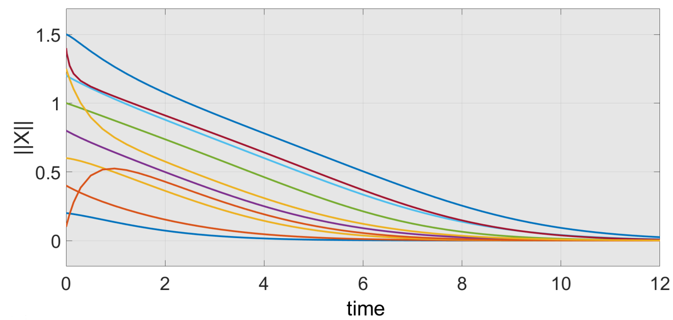

Fig. 3: Time simulations on Simulink.

V. Conclusions

In this paper, a comprehensive method has been introduced, which extends a previous published paper with the following added features: the Lyapunov function can be modeled by any differentiable function and a variety of stability problems are tackled. The result demonstrates the ability of the algorithm to determine a Lyapunov function modeled by any differentiable function while maximizing the domain of attraction. In the future, it is believed that the described approach could be used for complex and intelligent systems that we can find in industrial frameworks. Besides, future works deal with a model-free approach which can be achieved after identifying the system with a neural network.

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